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# About the stability of a $D4$-$\bar{D}4$ system [1]

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**Abstract**

We study a system of coincident $D4$ and $\bar{D}4$ branes with non zero world-volume magnetic fields in the weak coupling limit. We show that the conditions for absence of tachyons in the spectrum coincide exactly with those found in hep-th/0206041, in the low energy effective theory approach, for the system to preserve a $\frac{1}{4}$ of the supersymmetries of the Type IIA string theory vacuum. We present further evidence about the stability of the system by computing the lowest order interaction amplitude from both open and closed channels, thus verifying the no force condition as well as the supersymmetric character of the spectrum. A brief discussion of the low energy effective five dimensional world-volume theory is given.

## 1 Introduction

The discovery, in type II string theories, of cylinder-like branes preserving a quarter of the supersymmetries of the flat Minkowski space-time, the so-called "supertubes", has attracted much attention recently [4]-[8]. The stabilizing factor at the origin of their BPS character, which prevents them from collapsing, is the angular momentum generated by the non-zero gauge field that lives on the brane.

An interesting feature of the supertube is that it presents $D0$ and $F1$ charges, but no $D2$ charge. In relation to this fact, Bak and Karch (BK) observed that, if we consider the elliptical supertube in the limit when one of the semi-axis goes to infinity, the resulting system should be equivalent to having two flat 2-branes with total $D2$ charge equal to zero. This naturally led to conjecture the existence of SUSY $D2$-$\bar{D}2$ systems. Such a

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study as well as the study of systems with arbitrary numbers of $D2$ and $\bar{D}2$ branes was made in the context of the Born-Infeld action in reference [9], where the conditions to be satisfied by the Killing spinors were identified. Soon after that, in reference [10], higher dimensional brane-antibrane systems were considered in the Born-Infeld context (see [11] for related work in the matrix model context). In particular, the existence of a quarter SUSY $D4$-$\bar{D}4$ systems with $D2$ and Taub-NUT charges and no $D4$-brane charge that should represent genuine bound states of such components was conjectured. While it is plausible that a five dimensional supertube-like solution exists, leading in a certain limit to the brane-antibrane system, much as it happens with the supertube, in this paper we will focus on a detailed study of the conformal field theory and, in particular, in the absence of tachyonic instabilities in the system (in the supertube context such analysis was carried out in references [5], [12]).

# 2 Review of the construction of the $D4$-$\bar{D}4$ SUSY system

Let us start by remembering those results in [10] which are relevant to the subject studied in this paper.

Let us consider type $IIA$ superstring theory in the flat vacuum defined by the ten dimensional Minkowskian metric tensor $G = \eta_{MN} dX^M\, dX^N$, with constant dilaton and the other fields put to zero. This background preserves maximal SUSY, whose general Killing spinor is, in the standard local basis $dX^M$, a constant 32 dimensional Majorana spinor $\epsilon$. Let us consider a $D4$ (or $\bar{D}4$) brane, with world-volume coordinates $\{\xi^\mu, \mu = 0,1,2,3,4\}$, the embedding defined by $X^\mu(\xi) = \xi^\mu$, $\mu = 0,\ldots,4$ , $X^i(\xi) = 0$ , $i = 5,\ldots,9$, and an abelian gauge field $A_\mu(\xi) = \frac{1}{2}\, F_{\nu\mu}\, \xi^\nu$ living on the brane, being $F = dA$ the constant field strength. Then, the introduction of such a $D4$ brane in space-time will preserve the supersymmetries that satisfy [14] [3]

$$\Gamma\, \epsilon = \pm\, \epsilon \quad , \tag{2.1}$$

where the "-" sign on the r.h.s. corresponds to the $\bar{D}4$ brane with the *same* fields as the $D4$ brane because, by definition, it has opposite orientation to the $D4$ brane. This last orientation is defined by $\varepsilon_{01234} = +1$, which is present in the $\Gamma$-matrix [13] [4]

$$\begin{aligned} \Gamma &\equiv d^{-\frac{1}{2}} \left( \Gamma_{11} + \frac{1}{2} F_{\mu\nu}\, \Gamma^{\mu\nu} + \frac{1}{8} F_{\mu\nu}\, F_{\rho\sigma}\, \Gamma^{\mu\nu\rho\sigma}\, \Gamma_{11} \right) \Gamma^{01234} \\ d &\equiv \det d_+ = \det d_- > 0 \quad , \quad {d_\pm}^\mu{}_\nu = \delta^\mu{}_\nu + F^\mu{}_\nu \end{aligned} \tag{2.2}$$

[3] The scale $T_s = (2\,\pi\,\alpha')^{-1}$ is put to unity everywhere unless explicitly written.

[4] We take the ten dimensional $\Gamma$-matrices to obey $\{\Gamma^M ; \Gamma^N\} = 2\,\eta^{MN}$ , $\{\Gamma^M ; \Gamma_{11}\} = 0$ , with $\Gamma_{11} \equiv \Gamma^1 \ldots \Gamma^9\, \Gamma^0$. For definiteness, we adopt a Majorana-Weyl basis where $\Gamma^{M\,t} = \eta_{MM}\, \Gamma^M$ . In such basis, we can take $A_\pm = C_\pm$ , where $C_\pm\, \Gamma^M\, {C_\pm}^{-1} = \pm\Gamma^{M\dagger} = \pm\Gamma^{M\,t}$ defines the charge conjugation matrices.

Now, as discussed in [10], the anti-symmetric matrix of magnetic fields ($B_{ij} = F_{ij}$), with $i, j = 1, \ldots, 4$, can be put in the standard form $\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \otimes i\sigma_2$ by means of an $SO(4)$-rotation; the $SO(2) \times SO(2)$ rotation left over by this condition can then be used to put the electric field $E_i \equiv F_{i0}$ in the (13) plane. So we can consider, with no loss of generality

$$(F^{\mu}{}_{\nu}) = \begin{pmatrix} 0 & E_1 & 0 & E_3 & 0 \\ E_1 & 0 & B_1 & 0 & 0 \\ 0 & -B_1 & 0 & 0 & 0 \\ E_3 & 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & -B_2 & 0 \end{pmatrix} \quad , \tag{2.3}$$

because any other solution will be related to the ones with the configuration (2.3) by means of successive rotations.

One of the solutions found in [10] corresponds to a T-dual configuration of the Bak-Karch $D2 - \bar{D}2$ system; we will be interested at present on another one. This novel solution, which also preserves $\frac{1}{4}$ of SUSY, is obtained by restricting the field strength in the following way

$$B_1{}^2\, B_2{}^2 - B_1{}^2\, E_3{}^2 - B_2{}^2\, E_1{}^2 = 1 \tag{2.4}$$

This constraint clearly implies that $(B_{ij})$ cannot be singular; even more, if it holds, then the module of the four-vector $\vec{\beta} = -B^{-1}\, \vec{E}$ verifies

$$0 < \beta^2 = \left(\frac{E_1}{B_1}\right)^2 + \left(\frac{E_3}{B_2}\right)^2 = 1 - \frac{1}{B_1{}^2\, B_2{}^2} < 1 \quad . \tag{2.5}$$

It is possible to show that a boost with a velocity equal to $\vec{\beta}$ eliminates the electric field; a further rotation (which certainly does not affect the null electric field condition) can fix the magnetic fields matrix in the standard form again. So we conclude that the sector of fields obeying (2.4) is Lorentz-equivalent to a sector of the observers that do not see electric field and have non-singular magnetic matrix of determinant equal to one. Therefore, we will restrict our attention to a field strength (2.3) with

$$\begin{aligned} E_1 &= E_3 = 0 \\ \det B &= B_1{}^2\, B_2{}^2 = 1 \quad . \end{aligned} \tag{2.6}$$

Such field strength clearly breaks the space-time symmetry as follows

$$SO(1,9) \longrightarrow \mathbf{R} \times SO(2) \times SO(2) \times SO(5) \quad . \tag{2.7}$$

The $\Gamma$-matrix (2.2) takes the form

$$\begin{aligned} \Gamma &= d^{-\frac{1}{2}}\, (\Gamma_{01234}\, \Gamma_{11} + B_1\, \Gamma_{034} + B_2\, \Gamma_{012} + B_1\, B_2\, \Gamma_0\, \Gamma_{11}) \\ d &= 2 + B_1{}^2 + B_2{}^2 = (|B_1| + |B_2|)^2 \quad . \end{aligned} \tag{2.8}$$

The solution we are interested in is obtained by splitting equation (2.1) into the following two conditions

$$
\begin{aligned}
-\,B_1\,B_2\,\Gamma_{1234}\,\epsilon &= \epsilon \\
d^{-\frac{1}{2}}\,\left(B_1\,\Gamma_{034} + B_2\,\Gamma_{012}\right)\,\epsilon &= \pm\epsilon \quad .
\end{aligned} \tag{2.9}
$$

This system is consistent, and leads to the preservation of $\frac{1}{4}\,32 = 8$ SUSY charges; explicitly, with $-i\Gamma_{12}(s) = (-)^s\,(s)$ , etc (see [10] for details and notation), the 8 Killing spinors are

$$
\eta^{(\pm)}_{(s_1 s_2 s_3)} = (s s_1 s_2 s_3 1) \pm i\, sg(B_2)\,(-)^{\sum_{k=1}^{3} s_k}\,(s s_1 s_2 s_3 0) \quad , \quad (-)^s = sg(B_1\,B_2)\,(-)^{s_1} \ , \tag{2.10}
$$

where the labels $s$ take the values 0 (spin down) or 1 (spin up). From these equations we can see that the Killing spinors $\eta^{(-)}_{(s_1 s_2 s_3)}$ corresponding to the introduction of a $\bar{D}4$ brane with world-volume gauge fields $(-B_1, -B_2)$ are exactly the same as the Killing spinors $\eta^{(+)}_{(s_1 s_2 s_3)}$ of the $D4$-brane with fields $(B_1, B_2)$. Therefore, we are led to conjecture that the $\eta^{(+)}_{(s_1 s_2 s_3)}$ in (2.10) are Killing spinors (without taking into account back-reaction effects) of $\frac{1}{4}$ SUSY systems of $D4$ branes with fields $(B_1, B_2)$ and $\bar{D}4$ branes with the opposite ones $(-B_1, -B_2)$, obeying ${B_1}^2\,{B_2}^2 = 1$. In what follows, we will focus our attention on a $D4$-$\bar{D}4$ configuration.

# 3 The "light-cone" gauge-fixing and rotational invariance

As is well known, a covariant analysis of the string spectrum is carried out by considering the BRST charge [3]

$$
\begin{aligned}
Q^{BRST} \;\equiv\; & \sum_{m\in\mathbf{Z}} c_{-m}\,L^{(m)}_m + \sum_{r\in\mathbf{Z}_\delta} \gamma_{-r}\,G^{(m)}_r + \frac{1}{2}\sum_{m\in\mathbf{Z}} :c_{-m}\,L^{bc}_m: + \frac{\Delta^{bc}_0}{2}\,c_0 \\
&+ \sum_{m\in\mathbf{Z}} c_{-m}\,L^{\beta\gamma}_m - \sum_{m\in\mathbf{Z}}\sum_{r\in\mathbf{Z}_\delta} b_m\,\gamma_{r-m}\,\gamma_{-r} \quad ,
\end{aligned} \tag{3.1}
$$

where $(\beta$-$\gamma)$ $b$-$c$ are the ($\lambda = \frac{3}{2}$ super) $\lambda = 2$ ghost fields and the superscript "(m)" stands for "matter". It verifies

$$
\{Q^{BRST}; Q^{BRST}\} = \frac{c^{(m)} + c^{(g)}}{12}\left(\sum_{m\in\mathbf{Z}} m\,(m^2-1)\,c_{-m}\,c_m + \sum_{r\in\mathbf{Z}_\delta} (4r^2-1)\,\gamma_{-r}\,\gamma_r\right) \ . \tag{3.2}
$$

From (3.2), it follows that $Q^{BRST}$ is nilpotent iff the central charge of the matter system (whatever it is) is $c^{(m)} = -c^{(g)} = 15$ , where we have used (C.15). Physical states are then defined as cohomology classes of this operator. However for the sake of clarity we will analyze the spectrum in the light-cone gauge to be described in what follows.

Let us start with a brief review of the analogue of the light-cone gauge fixing procedure in the presence of branes, a subject which, to our understanding, is not covered deeply enough in the literature. Let us consider an open superstring theory that consists of a time-like coordinate $X^0$ with NN b.c., $d+1$ coordinates $X^I, I = 1, \ldots, d+1$ with homogeneous DD b.c. (fermionic partners are considered below) and an arbitrary $N = 1$ superconformal field theory. Let us pick up $X^0$ and $X^{d+1}$ to define "light-cone" coordinates $X^\pm \equiv X^0 \pm X^{d+1}$. Then, the b.c. are equivalently written as

$$\left.\begin{array}{r} \partial_\sigma X^0|_{\sigma=0} = \partial_\sigma X^0|_{\sigma=\pi} = 0 \\ \partial_\tau X^{d+1}|_{\sigma=0} = \partial_\tau X^{d+1}|_{\sigma=\pi} = 0 \end{array}\right\} \longleftrightarrow \left\{ \begin{array}{l} \partial_\pm X^+|_{\sigma=0} = \partial_\mp X^-|_{\sigma=0} \\ \partial_\pm X^+|_{\sigma=\pi} = \partial_\mp X^-|_{\sigma=\pi} \end{array}\right. \quad , \tag{3.3}$$

which show that, although it could seem at first sight a little bizarre to mix fields with different b.c., there is no obstacle to doing so, and the problem on the r.h.s of (3.3) is perfectly well-defined; the same happens if $X^{d+1}$ is ND or DN. What changes radically is the interpretation of the complete gauge fixing possible thanks to the (super) conformal invariance of the string theory. From (A.12), (A.14) we get for the light-cone coordinates

$$\begin{aligned} X^\pm(\tau,\sigma) &= x^\pm + X^\pm_L(\sigma^+) + X^\pm_R(\sigma^-) \\ X^\pm_L(\sigma^+) &= \alpha'\, p^\pm\, \sigma^+ + i\,\frac{l}{2} \sum_{m\in\mathbf{Z}'} \frac{\alpha^\pm_m}{m}\, e^{-im\sigma^+} \\ X^\pm_R(\sigma^-) &= \alpha'\, p^\mp\, \sigma^- + i\,\frac{l}{2} \sum_{m\in\mathbf{Z}'} \frac{\alpha^\mp_m}{m}\, e^{-im\sigma^-} \quad , \end{aligned} \tag{3.4}$$

where $x^\pm \equiv x^0 \pm x_0^{d+1} = x^0$ and $p^\pm \equiv p^0 \pm T_s\, \Delta x^{d+1} = p^0$. Now, in order to linearize the $L_0$ constraint, a reparameterization very different from the usual one (see for example [1]) must be considered. Let us try in fact, the one defined by

$$\begin{aligned} \tilde\tau(\tau,\sigma) &= \frac{1}{2\,\alpha'\,p^+}\,\Big(X^+_L(\sigma^+) + X^-_R(\sigma^-)\Big) = \tau + \frac{i}{l\,p^+} \sum_{m\in\mathbf{Z}'} \frac{\alpha^+_m}{m}\, e^{-im\tau}\ \cos m\sigma \\ \tilde\sigma(\tau,\sigma) &= \frac{1}{2\,\alpha'\,p^+}\,\Big(X^+_L(\sigma^+) - X^-_R(\sigma^-)\Big) = \sigma + \frac{1}{l\,p^+} \sum_{m\in\mathbf{Z}'} \frac{\alpha^+_m}{m}\, e^{-im\tau}\ \sin m\sigma \quad . \end{aligned} \tag{3.5}$$

This world-sheet diffeomorphism satisfies the crucial properties

- It is a conformal reparameterization;
- It preserves the region of the parameters $\tilde\tau \in \mathbf{R}\,,\ 0 \le \tilde\sigma \le \pi\,;$
- It leaves the b.c. (3.3) invariant.

In terms of these new parameters we get

$$
\begin{array}{rcl}
\tilde{X}^+(\tilde{\tau},\tilde{\sigma}) & \equiv & X^+(\tau,\sigma) = x^+ + \alpha'\, p^+\, \tilde{\sigma}^+ + \tilde{X}^+_R(\tilde{\sigma}^-) \\
\tilde{X}^-(\tilde{\tau},\tilde{\sigma}) & \equiv & X^-(\tau,\sigma) = x^- + \alpha'\, p^+\, \tilde{\sigma}^- + \tilde{X}^-_L(\tilde{\sigma}^+) \quad ,
\end{array} \tag{3.6}
$$

where

$$
\begin{array}{rcl}
\tilde{X}^+_R(\tilde{\sigma}^-) & \equiv & X^+_R(\sigma^-)|_{\sigma^-(\tilde{\sigma}^-)} = \alpha'\, \tilde{p}^-\, \tilde{\sigma}^- + i\, \dfrac{l}{2} \displaystyle\sum_{m\in\mathbf{Z}'} \dfrac{\tilde{\alpha}^-_m}{m}\, e^{-im\tilde{\sigma}^-} \\
\tilde{X}^-_L(\tilde{\sigma}^+) & \equiv & X^-_L(\sigma^+)|_{\sigma^+(\tilde{\sigma}^+)} = \alpha'\, \tilde{p}^-\, \tilde{\sigma}^+ + i\, \dfrac{l}{2} \displaystyle\sum_{m\in\mathbf{Z}'} \dfrac{\tilde{\alpha}^-_m}{m}\, e^{-im\tilde{\sigma}^+}
\end{array} \tag{3.7}
$$

define the new variables $\{\tilde{\alpha}^-_m\}$ in terms of the old ones $\{\alpha^-_m\}$. From (3.4) and (3.6) we see that the reparameterization just puts $\{\tilde{\alpha}^+_m = 0\ ,\ m \neq 0\}$ (in particular $\tilde{p}^- = p^- = p^0$) and lefts over a translation invariance in $\tilde{\tau}$, as in the usual case; however, $\tilde{X}^+$ *is not* the world-sheet time.

Analogously, for the fermionic superpartners $\psi^0$ and $\psi^{d+1}$ we introduce $\psi^\pm = \psi^0 \pm \psi^{d+1}$. But, while in the usual case we can reach the gauge $\psi^+ = 0$ through a superconformal transformation (A.5), it turns out that in our setting the gauge fixing allows to put $\psi^+_+ = \psi^-_- = 0 \leftrightarrow b^+_r = 0$ through a superconformal transformation defined by the parameters

$$
\epsilon^\pm(\sigma^\pm) = \mp \frac{\psi^\pm_\pm}{2\,\partial_\pm X^\pm} \quad . \tag{3.8}
$$

It is easy to check that this gauge fixing is compatible with (3.6). The super-Virasoro constraints are readily solved in this gauge; from (A.22), (A.24)

$$
\begin{array}{rcl}
T_{\pm\pm}(\sigma^\pm) & = & T^\perp_{\pm\pm}(\sigma^\pm) - \partial_\pm X^+\, \partial_\pm X^- = T^\perp_{\pm\pm}(\sigma^\pm) - \alpha'\, \dfrac{l\, p^+}{2} \displaystyle\sum_{m\in\mathbf{Z}} \alpha^-_m\, e^{-im\sigma^\pm} \\
G_\pm(\sigma^\pm) & = & G^\perp_\pm(\sigma^\pm) - \dfrac{1}{2}\, \psi^\mp_\pm\, \partial_\pm X^\pm = G^\perp_\pm(\sigma^\pm) - \dfrac{\alpha'\, l\, p^+}{2\sqrt{2}} \displaystyle\sum_{r\in\mathbf{Z}_\epsilon} b^-_r\, e^{-ir\sigma^\pm} \quad .
\end{array} \tag{3.9}
$$

It follows that the conditions $L_m - A\, \delta_{m,0} = G_r = 0$ yield

$$
\alpha^-_m = \frac{2}{l\, p^+}\, (L^\perp_m - A\, \delta_{m,0}) \quad , \quad b^-_r = \frac{2}{l\, p^+}\, G^\perp_r \tag{3.10}
$$

respectively. Here, $A$ is a normal ordering constant, while "$\perp$" stands for contributions other than the $(0, d+1)$ directions, i.e. the $(1, \ldots, d)$ directions and the $N = 1$ theory. In particular, $(L^\perp_m, G^\perp_r)$ generate an $N = 1$ superconformal algebra (A.26) with $c^\perp = \frac{3}{2}\, d + c_{N=1}$ .

This gauge fixing obscures the initial $SO(d+1)$ invariance of the system, leaving just the $SO(d)$ subgroup manifest. The tentative generators $\{J_{IJ}\} = \{J_{ij}, J_{i(d+1)} \equiv J_i\}$ in the gauge-fixed system are

$$
\begin{aligned}
J_{ij} &\equiv J^{(0)}_{ij} + J^{(b)}_{ij} + J^{(f)}_{ij} \\
&= \frac{1}{2\,i}\,[b^i_0\,;b^j_0\,] - i\sum_{m>0}\frac{1}{m}\,\left(\alpha^i_{-m}\,\alpha^j_m - \alpha^j_{-m}\,\alpha^i_m\right) - i\sum_{r>0}\left(b^i_{-r}\,b^j_r - b^j_{-r}\,b^i_r\right) \\
J_i &\equiv J^{(0)}_i + J^{(b)}_i + J^{(f)}_i \\
&= \frac{i}{l\,p^+}\,\left(b^i_0\,G^\perp_0 + \sum_{m>0}\frac{1}{m}\,\left(\alpha^i_{-m}\,L^\perp_m - L^\perp_{-m}\,\alpha^i_m\right) + \sum_{r>0}\left(b^i_{-r}\,G^\perp_r - G^\perp_{-r}\,b^i_r\right)\right)\quad ,
\end{aligned}
\tag{3.11}
$$

where $i, j = 1, \ldots, d$ and $J^{(0)}_{ij}$ and $J^{(0)}_i$ are absent in the NS sector. It is straightforward to show that

$$
[J_{ij}; J_{kl}] = -i\,(\delta_{il}\,J_{jk} + \delta_{jk}\,J_{il} - (i \leftrightarrow j)) \quad , \quad [J_{ij}; J_k] = i\,(\delta_{ik}\,J_j - \delta_{jk}\,J_i) \tag{3.12}
$$

while, as usual, the problematic commutator is $[J_i; J_j]$, whose careful computation gives

$$
\begin{aligned}
[J_i; J_j] &= i\,J_{ij}\,\frac{p^-}{p^+} + \frac{i}{(l\,p^+)^2}\,(2\,A - \frac{c^\perp}{12})\,J^{(0)}_{ij} \\
&+ \frac{1}{(lp^+)^2}\,\left(\sum_{m>0}\frac{\Delta^{(b)}_m}{m}\,\left(\alpha^i_{-m}\,\alpha^j_m - (i \leftrightarrow j)\right) + \sum_{r>0}\Delta^{(f)}_r\,\left(b^i_{-r}\,b^j_r - (i \leftrightarrow j)\right)\right) \quad .
\end{aligned}
\tag{3.13}
$$

The anomalies

$$
\Delta^{(b)}_m = \Delta^{(f)}_{\frac{m}{2}} = 2A - 1 + (\frac{c^\perp}{12} - 1)\,(m^2 - 1) \tag{3.14}
$$

show that the complete $SO(d+1)$ algebra closes *on-shell*, $p^+ = p^- = p^0$, in both sectors provided $c^\perp = 12$ and $A = \frac{1}{2}$ .

# 4 Analysis of the spectrum and supersymmetry

In this section, we will analyze the perturbative spectrum of the $D4$-$\bar{D}4$ system defined in Section 2. Therefore, we must consider open superstrings with a time-like NN coordinate $X^0$, four coordinates along the branes resumed in two complex fields $Z^{(1)} \equiv X^1 + iX^2, Z^{(2)} \equiv X^3 + iX^4$, and five DD coordinates $X^i, i = 5, \ldots, 9$ orthogonal to the branes. Each coordinate field is paired with fermionic partners $\psi^0$ (Majorana), $\Psi^{(1)}, \Psi^{(2)}$ (Dirac's) and $\psi^i, i = 5, \ldots, 9$ (Majoranas) respectively. Furthermore, we take from the start the magnetic fields on the $D4$ and $\bar{D}4$ branes to be $(B_1 = T_s\,\tan(\frac{\pi}{2}\nu^{(1)}), B_2 = T_s\,\tan(\frac{\pi}{2}\nu^{(2)}))$ and $(-B_1, -B_2)$ respectively, with $0 < |\nu^{(i)}| < 1$, but with no relation between the $B_i$ 's.

According to the precedent section, the theory obtained after fixing the light-cone like gauge in the (09) directions should present $SO(5)$ invariance, since $d = 4$ and, in fact,

$c^{\perp} = \frac{3}{2}\,4 + 6 = 12$. This invariance reflect in the spectrum, and we will give some examples below.

From (3.10) (with $A = \frac{1}{2}$), (A.22), (A.27), (B.4), (B.28) and (B.29), the energy operator in any sector reads

$$
\begin{aligned}
E^2 &\equiv (p^0)^2 = \frac{1}{\alpha'}\left(L_0^{\perp} - \frac{1}{2}\right) \\
&= \delta_{\nu_-^{(1)},0}\ \cos^2\varphi_0^{(1)}\ |\vec{p}^{(1)}|^2 + \delta_{\nu_-^{(2)},0}\ \cos^2\varphi_0^{(2)}\ |\vec{p}^{(2)}|^2 + \frac{1}{\alpha'}\left(N^{\perp} + N_0\right) \\
N_0 &= \Delta_0^{(1)} + \Delta_0^{(2)} + \sum_{i=5}^{8}\Delta_0^i - \frac{1}{2} = \delta\left(\bar{\nu}^{(1)} + \bar{\nu}^{(2)} - 1\right) \quad ,
\end{aligned}
\tag{4.1}
$$

where $N^{\perp} = N^{(1)} + N^{(2)} + \sum_{i=5}^{8} N^i$ is the total number operator in the eight transverse dimensions.

There are four “CP factors”, in the language of [22], which label the states in the spectrum, depending on where the ends of the (oriented) strings are fixed; we denote them by $dd, \bar{d}\bar{d}, d\bar{d}$ and $\bar{d}d$. The vacuum energy $N_0$ depends on the CP label and on whether we are in the NS sector ($\delta = \frac{1}{2}$) or in R sector ($\delta = 0$). Furthermore, the spectrum must be GSO projected, procedure which preserves the states such that

$$
(-)^{\sharp + \nu_-^{(1)} - \bar{\nu}^{(1)} + \nu_-^{(2)} - \bar{\nu}^{(2)}} \left\{ \begin{array}{l} 1 \\ \gamma \end{array} \right. = +1 \tag{4.2}
$$

in the NS and R sectors respectively. Here, $\sharp$ stands for the number of fermionic oscillators and $\gamma \equiv -4\, b_0^5\, b_0^6\, b_0^7\, b_0^8$ is the chirality operator of $spin(4)$, $\{b_0^i; b_0^j\} = \delta^{ij}$ being the Clifford algebra. The justification for this rule is given in the next Section, equation (5.8).

Let us look first at the lowest levels. In the $dd$ sector the “massless” (null energy) spectrum is the well-known vector multiplet corresponding to a theory with 16 supercharges, consisting of a a five dimensional $U(1)$ vector field, five scalars and the corresponding fermions; idem in the $\bar{d}\bar{d}$ sector. The $d\bar{d}$ superstrings, on the other hand, break half of this supersymmetry, but the states which survive GSO are different in the following cases

**Case** $b^{(1)} b^{(2)} > 0$

$$
\begin{array}{rl}
|0>_{NS} & , \quad \alpha' E^2 = \dfrac{sign\, b^{(1)}}{2}\left(|\nu_-^{(1)}| + |\nu_-^{(2)}| - 1\right) \\
B^{(1)}_{\bar{\nu}^{(1)} - \frac{1}{2}}\, B^{(2)}_{\bar{\nu}^{(2)} - \frac{1}{2}}\, |0>_{NS} & , \quad \alpha' E^2 = \dfrac{sign\, b^{(1)}}{2}\left(1 - |\nu_-^{(1)}| - |\nu_-^{(2)}|\right) \\
P_+|\alpha> & , \quad \alpha' E^2 = 0 \quad , \quad \alpha = 1, \ldots, 4
\end{array}
\tag{4.3}
$$

**Case** $b^{(1)} b^{(2)} < 0$

$$
\begin{array}{rl}
B^{(a)}_{\bar{\nu}^{(a)} - \frac{1}{2}}\, |0>_{NS} & , \quad E^2 = \dfrac{sign\, b^{(a)}}{2\alpha'}\left(1 - |\nu_-^{(1)}| - |\nu_-^{(2)}|\right) \\
P_-|\alpha> & , \quad \alpha' E^2 = 0 \quad , \quad \alpha = 1, \ldots, 4
\end{array}
\tag{4.4}
$$

where the states in the last lines are in the R sector, $|\alpha>$ being the spinor representation of $spin(4)$ algebra and $P_\pm \equiv \frac{1}{2}(1 \pm \gamma)$. We see that, under the condition (2.6), the potential tachyons disappear and the NS sector contributes to the massless level with two complex fields (considering the $\bar{d}d$ superstrings). These, together with the fermions, fill a $4+4=8$ dimensional representation of a superalgebra with eight supercharges, corresponding to a massless hypermultiplet [3, 20].

Let us make a further step and write down the first two massive levels (in the inter-brane sector) indicating, in the first (second) column, the NS (R) sector states.

**Case** $b^{(1)} > 0\,,\ b^{(2)} > 0$

- $\alpha'\, E^2 = |\nu_-^{(1)}| = \bar{\nu}^{(1)} = 1 - \bar{\nu}^{(2)}$

$$
\begin{array}{rc}
A^{(1)\,\dagger}_{\bar{\nu}^{(1)}}\,|0>_{NS} & \\
A^{(2)}_{\bar{\nu}^{(2)}-1}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}|0>_{NS} & A^{(a)\,\dagger}_{\bar{\nu}^{(a)}}\,P_+|\alpha> \\
\left(A^{(1)\,\dagger}_{\bar{\nu}^{(1)}}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}} - A^{(2)}_{\bar{\nu}^{(2)}-1}\right)|0>_{NS} & B^{(a)\,\dagger}_{\bar{\nu}^{(a)}}\,P_-|\alpha> \\
B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}\,b^i_{-\frac{1}{2}}\,|0>_{NS}\ ,\ A^{(2)}_{\bar{\nu}^{(2)}-1}\,|0>_{NS} &
\end{array}
\tag{4.5}
$$

- $\alpha'\, E^2 = |\nu_-^{(2)}| = \bar{\nu}^{(2)} = 1 - \bar{\nu}^{(1)}$

$$
\begin{array}{rc}
A^{(2)\,\dagger}_{\bar{\nu}^{(2)}}\,|0>_{NS} & \\
A^{(1)}_{\bar{\nu}^{(1)}-1}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}|0>_{NS} & A^{(a)}_{\bar{\nu}^{(a)}-1}\,P_+|\alpha> \\
\left(A^{(2)\,\dagger}_{\bar{\nu}^{(2)}}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}} - A^{(1)}_{\bar{\nu}^{(1)}-1}\right)|0>_{NS} & B^{(a)}_{\bar{\nu}^{(a)}-1}P_-|\alpha> \\
B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,b^i_{-\frac{1}{2}}\,|0>_{NS}\quad,\quad A^{(1)}_{\bar{\nu}^{(1)}-1}\,|0>_{NS} &
\end{array}
\tag{4.6}
$$

**Case** $b^{(1)} < 0\,,\ b^{(2)} < 0$

As the precedent one, with the exchanging of levels $|\nu_-^{(1)}| \leftrightarrow |\nu_-^{(2)}|$.

**Case** $b^{(1)} > 0\,,\ b^{(2)} < 0$

- $\alpha'\, E^2 = |\nu_-^{(1)}| = \bar{\nu}^{(1)} = \bar{\nu}^{(2)}$

$$
\begin{array}{rc}
A^{(1)\,\dagger}_{\bar{\nu}^{(1)}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}\,|0>_{NS} & \\
A^{(2)\,\dagger}_{\bar{\nu}^{(2)}}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}}\,|0>_{NS} & A^{(a)\,\dagger}_{\bar{\nu}^{(a)}}\,P_-|\alpha> \\
\left(A^{(1)\,\dagger}_{\bar{\nu}^{(1)}}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}} - A^{(2)\,\dagger}_{\bar{\nu}^{(2)}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}\right)|0>_{NS} & B^{(a)\,\dagger}_{\bar{\nu}^{(a)}}\,P_+|\alpha> \\
b^i_{-\frac{1}{2}}\,|0>_{NS}\quad,\quad \left(A^{(1)\,\dagger}_{\bar{\nu}^{(1)}}\,B^{(1)}_{\bar{\nu}^{(1)}-\frac{1}{2}} + A^{(2)\,\dagger}_{\bar{\nu}^{(2)}}\,B^{(2)}_{\bar{\nu}^{(2)}-\frac{1}{2}}\right)|0>_{NS} &
\end{array}
\tag{4.7}
$$

- $\alpha' E^2 = |\nu_-^{(2)}| = 1 - \bar\nu^{(2)} = 1 - \bar\nu^{(1)}$

$$
\begin{array}{ll}
 & A^{(1)}_{\bar\nu^{(1)}-1}\, B^{(2)}_{\bar\nu^{(2)}-\frac{1}{2}}\,|0>_{NS} \\
 & A^{(2)}_{\bar\nu^{(2)}-1}\, B^{(1)}_{\bar\nu^{(1)}-\frac{1}{2}}\,|0>_{NS} \\
 & \left(A^{(1)}_{\bar\nu^{(1)}-1}\, B^{(1)}_{\bar\nu^{(1)}-\frac{1}{2}} - A^{(2)}_{\bar\nu^{(2)}-1}\, B^{(2)}_{\bar\nu^{(2)}-\frac{1}{2}}\right)|0>_{NS} \\
b^i_{-\frac{1}{2}}\,|0>_{NS}\ \ , & \left(A^{(1)}_{\bar\nu^{(1)}-1}\, B^{(1)}_{\bar\nu^{(1)}-\frac{1}{2}} + A^{(2)}_{\bar\nu^{(2)}-1}\, B^{(2)}_{\bar\nu^{(2)}-\frac{1}{2}}\right)|0>_{NS}
\end{array}
\qquad
\begin{array}{l}
A^{(a)}_{\bar\nu^{(a)}-1} P_- |\alpha> \\
B^{(a)}_{\bar\nu^{(a)}-1} P_+ |\alpha>
\end{array}
\tag{4.8}
$$

**Case** $b^{(1)} > 0\ ,\ b^{(2)} < 0$

As the precedent one, with the exchanging of levels $|\nu_-^{(1)}| \leftrightarrow |\nu_-^{(2)}|$.

In each case and level, we have arranged the spectrum in such a way that the three NS states in the first three lines are $SO(5)$ scalars, while the five states in the last line form an $SO(5)$ vector; in the R sector, each value of $a = 1, 2$ labels a $Spin(5)$ Dirac spinor. These assertions can be easily checked by applying (3.11) on the states and, of course, they signal the $SO(5)$ invariance of the spectrum. In any case, the states in each level expand a $16 + 16 = 32$ dimensional representation of a superalgebra with eight supercharges, corresponding to a massive (non BPS) supermultiplet [3, 20].

# 5 Vacuum amplitudes and boundary states

## 5.1 The open string channel

Let us consider a $Dp$-brane located at $X^i = y_0^i\,,\ i = p+1, \ldots, D-1$ and another one (or a $\bar{D}p$-brane) at $X^i = y_\pi^i\,,\ i = p+1, \ldots, D-1$ in $D$-dimensional flat space. By means of a straightforward generalization of (B.1), (B.2) for arbitrary constant field strengths ${F_0}^\mu{}_\nu = T_s\, {f_0}^\mu{}_\nu$ and ${F_\pi}^\mu{}_\nu = T_s\, {f_\pi}^\mu{}_\nu$ living on their world-volumes along the $\mu = 0, 1, \ldots, p$ directions, the b.c. for the coordinate fields of open strings suspended between them result

$$
\partial_\sigma X^\mu(\tau, 0) - {f_0}^\mu{}_\nu\, \partial_\tau X^\nu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) - {f_\pi}^\mu{}_\nu\, \partial_\tau X^\nu(\tau, \pi) = 0 \quad . \tag{5.1}
$$

The *one-loop* interaction diagram is constructed by imposing conditions of periodicity (P), $\eta_\pm = +1$, or anti-periodicity (AP), $\eta_\pm = -1$, in the euclidean time variable $\tau^e \equiv i\tau \sim \tau^e + T$

$$
X^M(\tau, \sigma) \sim X^M(\tau - iT, \sigma) \quad , \quad \psi^M_\pm(\tau, \sigma) \sim \eta_\pm\, \psi^M_\pm(\tau - iT, \sigma) \quad . \tag{5.2}
$$

This is carried out by taking the traces in the Hilbert space, remembering that, in the case of fermions and ghost system with P b.c., we must insert the spinor number operator (written for convenience in pairs of indices, see (B.26))

$$
(-)^{F^\Psi} = \prod_{a=1}^{5} (-)^{-J_0^{\Psi\,(a)}} \quad . \tag{5.3}
$$

The $\lambda = 2$ ghost fields $b$-$c$ and $\lambda = \frac{3}{2}$ superghost fields $\beta$-$\gamma$ follow the b.c. of the (bosonic) reparameterization and (fermionic) SUGRA transformations parameters respectively. The insertion of the spinor number operators

$$(-)^{F^{bc}} = (-)^{U_0^{bc}} \quad , \quad (-)^{F^{\beta\gamma}} = (-)^{U_0^{\beta\gamma}} \tag{5.4}$$

must be carried out when P (AP) b.c. apply, due to the fermionic (bosonic) character of the ghost (superghost) system; the definition of the $U_0$ charges is given in (C.8).

The connected part of the one loop amplitude is guessed from the Coleman-Weinberg formula

$$A^{1-loop} \equiv \ln\ Z^{1-loop} \sim -\frac{1}{2}\, tr\ (-)^{\mathbf{F}} \ln\ G^{-1} \tag{5.5}$$

where $G^{-1} = p^2 + M^2 = \alpha'^{-1} L_0$ is the inverse (free) propagator, $\mathbf{F}$ is the *space-time* fermion number and the traces are on the full Hilbert space. Regulating as usual the logarithm, we define

$$\begin{aligned} \mathcal{A}^{open} &= -\frac{1}{2}\, tr_{NS}\ \ln \frac{G^{-1}}{T_s} + \frac{1}{2}\, tr_R\ \ln \frac{G^{-1}}{T_s} = \int_0^\infty \frac{dt}{2\,t}\ (A_{NS}^{open}(it) + A_R^{open}(it)) \\ A_{NS,R}^{open}(\tau) &= tr_{NS,R}\ q^{L_0}\ (-)^{F^\Psi + F^{bc}}|_{q=e^{i2\pi\tau}} = A_{open}^{(b)}(\tau)\ A_{open}^{(f)}(\tau)|_{NS,R} \quad . \end{aligned} \tag{5.6}$$

In what follows, we focus on our system. The bosonic contribution is

$$\begin{aligned} A_{open}^{(b)}(it) &\equiv Z^{X^0}(\tau) \prod_{i=5}^{9} Z^{X^i}(\tau)\ Z^{Z^{(1)}}(\tau)\ Z^{Z^{(2)}}(\tau)\ Z^{bc}(\tau) \\ &= i\,V_5\ e^{i\pi(1-\bar\nu^{(1)}-\bar\nu^{(2)})} \frac{16\, b^{(1)}\, b^{(2)}}{(8\pi^2\alpha')^{\frac{5}{2}}}\ \frac{e^{-T_s\,\Delta\vec{y}^2\,t}}{t^{\frac{1}{2}}\ \eta(it)^4}\ \left(Z_1^{1-2\bar\nu^{(1)}}(it)\ Z_1^{1-2\bar\nu^{(2)}}(it)\right)^{-1} \quad , \end{aligned} \tag{5.7}$$

where we have used the formulae given in (D.3). [5]

In the fermionic sector, we must impose the GSO condition which in the presence of non trivial b.c., becomes a little bit subtle. It is equivalent to inserting in the traces the projection operator

$$P_{GSO} \equiv \frac{1}{2}\ \left(1 - (-)^{\nu_-^{(1)}+\nu_-^{(2)}}\ (-)^{F^\Psi + F^{\beta\gamma}}\right) \quad . \tag{5.8}$$

The logic for this definition relies in two facts,

- ${P_{GSO}}^2 = P_{GSO}$ must hold in the Hilbert space of the (perturbative) theory;
- On physical grounds it should reduce to the well-known operator

  $$P_{GSO} \equiv \frac{1}{2}\ \left(1 - (-)^{F^\Psi + F^{\beta\gamma}}\right) \tag{5.9}$$

  if we turn off adiabatically the gauge fields (note that the definition is in terms of $\nu_-^{(a)}$, *not* of $\bar\nu^{(a)}$, see (B.4); note also the sign flipped w.r.t. a *Dp*-*Dp* system [22]).

[5] In $Z^{bc}$ the zero mode sector must be projected out [2].

Carrying out the computations we get

$$
\begin{aligned}
A^{(f)}_{open}(it) &= A^{(f);GSO}_{open}(it)|_{NS} + A^{(f);GSO}_{open}(it)|_{R} \\
A^{(f);GSO}_{open}(it)|_{NS} &\equiv tr_{NS} \prod_{a=1}^{5} q^{L_0^{\Psi^{(a)}}-\frac{1}{24}} \, (-)^{F^{\Psi}+\sum_{a=1}^{5} q_0^{(a)}} \, q^{L_0^{\beta\gamma}-\frac{11}{24}} \, (-)^{[\pi_0]} \, P_{GSO} \\
&= e^{i\pi(\bar\nu^{(1)}+\bar\nu^{(2)})} \, \frac{1}{2} \Big( Z_1^0(it)^2 \, Z_1^{-2\nu_-^{(1)}}(it) \, Z_1^{-2\nu_-^{(2)}}(it) \\
&+ \; e^{-i\pi(\nu_-^{(1)}+\nu_-^{(2)})} \, Z_0^0(it)^2 \, Z_0^{-2\nu_-^{(1)}}(it) \, Z_0^{-2\nu_-^{(2)}}(it) \Big) \\
A^{(f);GSO}_{open}(it)|_{R} &\equiv tr_{R} \prod_{a=1}^{5} q^{L_0^{\Psi^{(a)}}-\frac{1}{24}} \, (-)^{F^{\Psi}+\sum_{a=1}^{5} q_0^{(a)}} \, q^{L_0^{\beta\gamma}-\frac{11}{24}} \, (-)^{[\pi_0]} \, P_{GSO} \\
&= e^{i\pi(\bar\nu^{(1)}+\bar\nu^{(2)})} \, \frac{1}{2} \Big( -Z_1^1(it)^2 \, Z_1^{1-2\nu_-^{(1)}}(it) \, Z_1^{1-2\nu_-^{(2)}}(it) \\
&- \; e^{-i\pi(\nu_-^{(1)}+\nu_-^{(2)})} \, Z_0^1(it)^2 \, Z_0^{1-2\nu_-^{(1)}}(it) \, Z_0^{1-2\nu_-^{(2)}}(it) \Big) \quad . \qquad (5.10)
\end{aligned}
$$

From (5.6), (5.7), (5.10), we obtain the final result [6]

$$
\begin{aligned}
&\mathcal{A}^{open} = -i\, V_5 \, \frac{32\, b_0^{(1)}\, b_0^{(2)}}{(8\pi^2\alpha')^{\frac{5}{2}}} \int_0^\infty \frac{dt}{2\,t} \, \frac{e^{-T_s\,\Delta\vec y^2\, t}}{t^{\frac{1}{2}}\,\eta(it)^4} \left( Z_1^{1-2\nu_-^{(1)}}(it)\, Z_1^{1-2\nu_-^{(2)}}(it) \right)^{-1} \\
&\frac{1}{2} \Big( Z_1^0(it)^2 \, Z_1^{-2\nu_-^{(1)}}(it) \, Z_1^{-2\nu_-^{(2)}}(it) + e^{-i\pi(\nu_-^{(1)}+\nu_-^{(2)})} \, Z_0^0(it)^2 \, Z_0^{-2\nu_-^{(1)}}(it) \, Z_0^{-2\nu_-^{(2)}}(it) \\
&-Z_1^1(it)^2 \, Z_1^{1-2\nu_-^{(1)}}(it) \, Z_1^{1-2\nu_-^{(2)}}(it) - e^{-i\pi(\nu_-^{(1)}+\nu_-^{(2)})} \, Z_0^1(it)^2 \, Z_0^{1-2\nu_-^{(1)}}(it) \, Z_0^{1-2\nu_-^{(2)}}(it) \Big) \quad . \qquad (5.11)
\end{aligned}
$$

## 5.2 The closed string channel

The open string channel description of the interaction between branes just given has a dual closed string channel picture as exchange of *closed* strings between the so called "boundary states" (b.s.) $|B>$, that represent each brane as a sort of condensate of closed strings (see [15] for a review in the covariant formalism, which we will adopt; for a light-cone approach, see [16]). They are completely determined, up to normalization, by conditions that can be guessed by considering the conformal map

$$
w \equiv \tau^e + i\sigma = i\sigma^+ \longrightarrow \hat w = -i\,\frac{\pi}{T}\, w = \hat\tau^e + i\hat\sigma = i\hat\sigma^+ \Longleftrightarrow \hat\sigma^\pm = \mp i \frac{\pi}{T}\,\sigma^\pm \quad . \qquad (5.12)
$$

Clearly, the coordinate $\hat\sigma = -\frac{\pi}{T}\tau^e \sim \hat\sigma + \pi$ becomes the (periodic) spatial coordinate of the closed string, while $\hat\tau^e = \frac{\pi}{T}\sigma \in [0, \hat T = \frac{\pi^2}{T}]$. This draws the *tree level* cylinder interaction

[6] The bosonic and fermionic amplitudes are separately invariant under $\nu_-^{(a)} \to -\nu_-^{(a)}$; it follows that the contribution from exchanging the ends of the strings is taken into account just with the introduction of a factor of two, as was made in (5.11).

diagram between a b.s. at $\hat{\tau} = 0$ and a b.s. at $\hat{\tau} = \hat{T}$, dual to the one-loop open string diagram just computed.

Under this conformal map general $\lambda$-tensors transform as

$$t_{\pm}^{(\lambda)} d^{\lambda} \sigma^{\pm} = \hat{t}_{\pm}^{(\lambda)} d^{\lambda} \hat{\sigma}^{\pm} \quad \Longleftrightarrow \quad \hat{t}_{\pm}^{(\lambda)} = \left( \mp \frac{i\,\pi}{T} \right)^{-\lambda} t_{\pm}^{(\lambda)} \quad . \tag{5.13}$$

This relation is crucial in order to get the right b.s. definition. While, in the $\hat{\sigma}$- coordinate, the P (or AP) conditions just give the usual closed string expansions, [7] the conditions on $\hat{\tau}^e$ (coming from the $\sigma$ conditions in the open string picture) are interpreted as operator equations to be satisfied by the b.s. $|B>$. The amplitude between two branes is then given by a closed superstring matrix element

$$\begin{aligned} \mathcal{A}^{closed} &\equiv \; < B'|\, G\, \delta\left(L_0 - \tilde{L}_0\right)\, |B> = \frac{\alpha'}{4\,\pi} \int_{|z|<1} \frac{d^2 z}{z\, \bar{z}} \; < B'|\, z^{L_0}\; \bar{z}^{\tilde{L}_0} \;\, |B> \\ &= \int_0^{\infty} \frac{dt}{2\,t}\, \pi\, \alpha'\, t\, A^{closed}(it) \quad , \end{aligned} \tag{5.14}$$

where $L_0$ and $\tilde{L}_0$ are the total zero mode Virasoro generators in the left and right sectors and $G \equiv (p^2 + M^2)^{-1} = \frac{\alpha'}{2}\,(L_0 + \tilde{L}_0)^{-1}$ is the closed superstring propagator. Moreover from the factorization of the b.s.

$$A^{closed}(it) = A^X(z,\tilde{z})\; A^{bc}(z,\tilde{z})\; A^{\psi}(z,\tilde{z})\; A^{\beta\gamma}(z,\tilde{z})|_{z=\bar{z}=e^{-\pi\, t}} = A^{(b)}_{closed}(it)\; A^{(f)}_{closed}(it) \quad . \tag{5.15}$$

We remark that the "bra" b.s. must be defined in such a way that for any "ket" $|\psi>$, $< B|\psi > \equiv (|B>; |\psi>)$ for a hermitian scalar product $< \xi|\psi >^{*} = < \psi|\xi >$ which respects the hermiticity conditions of the fields under consideration.

We resume below the b.s. at $\tau = 0$ (we throw away the subindex "0") [8] as well as the amplitudes $A(z,\tilde{z})'s$ for each field.

<u>Scalar sector boundary state</u>

The b.c. (5.1) for the scalar periodic fields $X^M$ become, in the closed channel,

$$\begin{aligned} \left( \partial_{\hat{\sigma}^+} \hat{X}^{\mu}(0,\hat{\sigma}) + S^{\mu}{}_{\nu}\; \partial_{\hat{\sigma}^-} \hat{X}^{\nu}(0,\hat{\sigma}) \right)\; |B^X> &= 0 \qquad , \quad \mu,\nu = 0,1,\ldots p \\ \hat{X}^i(0,\hat{\sigma})\; |B^X> &= y_0^i\; |B^X> \quad , \quad i = p+1,\ldots,p+d_{\perp} \;\; , \end{aligned} \tag{5.16}$$

where $S \equiv (d^-)^{-1}\, d^+$ , $d^{\pm} = 1 \pm f$ and $d_{\perp} = D - 1 - p$. Equivalently, in terms of modes, these conditions translate to

[7] We follow the conventions of reference [1].

[8] It is evident, from (5.14) and the fact that the evolution operator is $H = L_0 + \tilde{L}_0$, that the amplitude is independent of the value of $\tau$ at which we compute it.

$$
\begin{aligned}
p^\mu \,|B^X> &= 0 \\
x^i \,|B^X> &= y_0^i \,|B^X> \\
\left(\alpha_m^M + M^M{}_N \,\tilde{\alpha}_{-m}^N\right) |B^X> &= 0 \quad , \quad m \neq 0 \quad ,
\end{aligned} \tag{5.17}
$$

where $M \equiv \begin{pmatrix} S^{-1} & 0 \\ 0 & -1_{d_\perp} \end{pmatrix}$. The solution for the b.s. defined by (5.16) is

$$
\begin{aligned}
|B^X> &= N_B \prod_{m=1}^{\infty} e^{-\frac{1}{m}\alpha_{-m}^M \, M_{MN} \, \tilde{\alpha}_{-m}^N} \,|B^X>_0 \\
|B^X>_0 &= |p^\mu = 0; x^i = y_0^i > \otimes |0> \quad ,
\end{aligned} \tag{5.18}
$$

where we have attached to it the normalization constant $N_B$. The amplitude is computed by using the usual pairing defined by the hermiticity conditions $\alpha_m^{M\dagger} = \alpha_{-m}^M$ , $\tilde{\alpha}_m^{M\dagger} = \tilde{\alpha}_{-m}^M$ for the left and right oscillators and $<0|0> \equiv 1$ , $<p'|p> \equiv \delta^D(p-p')$ in the oscillator and zero mode sectors respectively. We get

$$
\begin{aligned}
& A^X(z,\tilde{z}) \equiv < B'^X| \, z^{L_0^X - \frac{10}{24}} \, \tilde{z}^{\tilde{L}_0^X - \frac{10}{24}} \, |B^X> \\
= \;& N_B'^{\,*} \, N_B \, V_{p+1} \left(\frac{T_s}{\log|z|^{-1}}\right)^{\frac{d_\perp}{2}} \frac{e^{-\frac{\pi T_s \Delta y^2}{\log|z|^{-1}}}}{|z|^{\frac{5}{6}}} \prod_{m=1}^{\infty} \det\left(1_D - |z|^{2m} \, M^{-1} \, M'\right)^{-1} \quad ,
\end{aligned} \tag{5.19}
$$

where $\Delta y = \sqrt{(\vec{y}_0' - \vec{y}_0)^2}$ is the separation between branes (to be taken to zero).

<u>*b*-*c* ghost boundary state</u>

It is straightforward to see, with the help of (5.13), that the open string b.c. (C.2) of a $\lambda = 2$ anticommuting, periodic *b*-*c* system translate, in the closed channel, into the following conditions for the b.s. $|B^{bc}>$

$$
\begin{aligned}
\left(\hat{b}(0,\hat{\sigma}) - \hat{\tilde{b}}(0,\hat{\sigma})\right) |B^{bc}> &= 0 \quad \longleftrightarrow \quad \left(\hat{b}_m - \hat{\tilde{b}}_{-m}\right) |B^{bc}> = 0 \\
\left(\hat{c}(0,\hat{\sigma}) + \hat{\tilde{c}}(0,\hat{\sigma})\right) |B^{bc}> &= 0 \quad \longleftrightarrow \quad \left(\hat{c}_m + \hat{\tilde{c}}_{-m}\right) |B^{bc}> = 0 \ ,
\end{aligned} \tag{5.20}
$$

for any $m \in \mathbf{Z}$. The b.s. that solves (5.20) is [9]

$$
\begin{aligned}
|B^{bc}> &= \prod_{m=1}^{\infty} e^{c_{-m}\tilde{b}_{-m} - b_{-m}\tilde{c}_{-m}} \, |B^{bc}>_0 \\
|B^{bc}>_0 &= \frac{1}{\sqrt{2}} \left( |+-> - |-+> \right) \quad .
\end{aligned}
\tag{5.24}
$$

The hermiticity conditions, ${b_m}^\dagger = b_{-m}$ , ${c_m}^\dagger = c_{-m}$ , ${\tilde{b}_m}^\dagger = \tilde{b}_{-m}$ , ${\tilde{c}_m}^\dagger = \tilde{c}_{-m}$ , impose that the only non-zero pairings are

$$
<+-|-+> = - <-+|+-> = i \quad . \tag{5.25}
$$

In particular, ${}_0 < B^{bc}|B^{bc}>_0 = \frac{1}{2}\,(-i+i) = 0$ ; however, the amplitude is

$$
A^{bc}(z,\bar{z}) \equiv < B^{bc}|(b_0 + \tilde{b}_0)\,(c_0 - \tilde{c}_0)\; z^{L_0^{bc} + \frac{26}{24}}\; \bar{z}^{\tilde{L}_0^{bc} + \frac{26}{24}}\; |B^{bc}> = i\,|z|^{\frac{1}{6}} \prod_{m=1}^{\infty} \left(1 - |z|^{2m}\right)^2 \quad , \tag{5.26}
$$

where the zero mode insertions coming from the measure (taken into account in (5.7)) are translated according to (5.13), (C.4) [2].

Fermionic boundary state

We will heavily rely on superconformal invariance to carry out the analysis. Given $D = 10$ Majorana fermions, the open string boundary term (A.7)

$$
i\,\frac{T_s}{2} \int d\tau\; \eta_{MM}\; (\delta\psi_+^M\; \psi_+^N - \delta\psi_-^M\; \psi_-^N)|_{\sigma=0}^{\sigma=\pi} \tag{5.27}
$$

cancels when the following b.c. hold

$$
\psi_-^M(0,\sigma) = {\Lambda_0}^M{}_N\; \psi_+^N(0,\sigma) \quad , \quad \psi_-^M(\pi,\sigma) = {\Lambda_\pi}^M{}_N\; \psi_+^N(\pi,\sigma) \quad , \tag{5.28}
$$

where $\Lambda_0$ and $\Lambda_\pi$ are arbitrary $O(1,9)$ matrices. However, once we embed this theory into a superstring one by adding $D = 10$ bosonic partners with b.c. (5.1), compatibility with the superconformal transformations (A.5) fix them almost uniquely to be

$$
\Lambda_0 = \eta_0\; M_0 \quad , \quad \Lambda_\pi = \eta_\pi\; M_\pi \quad , \quad {\eta_0}^2 = {\eta_\pi}^2 = 1 \quad , \tag{5.29}
$$

[9] The zero mode ghost sector is realized by defining the four states

$$
|s\tilde{s}> \equiv |s> \otimes |\tilde{s}> \quad , \quad s,\tilde{s} = +,- \quad , \tag{5.21}
$$

on which the zero mode operators act as

$$
c_0 = \sigma_+ \otimes 1_2 \quad , \quad b_0 = \sigma_- \otimes 1_2 \quad , \quad \tilde{c}_0 = \sigma_3 \otimes \sigma_+ \quad , \quad \tilde{b}_0 = \sigma_3 \otimes \sigma_- \quad , \tag{5.22}
$$

where $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and $\sigma_i$ are the Pauli matrices. As usual, for the other operators

$$
b_m|s\tilde{s}> = \tilde{b}_m|s\tilde{s}> = c_{-m}|s\tilde{s}> = \tilde{c}_{-m}|s\tilde{s}> = 0 \quad , \quad m = 1,2,\ldots \quad . \tag{5.23}
$$

together with the periodicity of the SUSY parameter

$$\epsilon^+(\tau) = -\eta_0\, \epsilon^-(\tau) = \eta_0\, \eta_\pi\, \epsilon^+(\tau + 2\,\pi) \quad \longleftrightarrow \quad \epsilon^+(\sigma^+)|_{\sigma=0,\pi} = -\eta_{0,\pi}\, \epsilon^-(\sigma^-)|_{\sigma=0,\pi} \quad . \tag{5.30}$$

It is worth to note that the matrices $M_{0,\pi}$ (as defined below (5.17)) belong, in fact, to $SO(1,p) \subset O(1,9)$, as it can be readily checked.

Around the loop, the fermions can be P or AP, according to (5.2); from (5.13) (with $\lambda = \frac{1}{2}$) they become, in the closed channel,

$$\hat{\psi}^M_\pm(\hat{\tau}, \hat{\sigma} + \pi) = \eta_\pm\, \hat{\psi}^M_\pm(\hat{\tau}, \hat{\sigma}) \quad , \quad {\eta_\pm}^2 = 1 \quad , \tag{5.31}$$

giving rise to the well-known four sectors of the closed string; R-R if $\eta_+ = \eta_- = +1$, NS-NS if $\eta_+ = \eta_- = -1$, NS-R if $\eta_+ = -\eta_- = +1$, and R-NS if $\eta_+ = -\eta_- = -1$. Then, from (5.13), (5.28) and (5.29) (with $\eta_0 \equiv \eta$), we get the defining relations for the fermionic b.s.

$$\left(\hat{\psi}^M_-(0, \hat{\sigma}) - i\,\eta\; M^M{}_N\; \hat{\psi}^N_+(0, \hat{\sigma})\right)\; |B^\psi; \eta >= 0 \quad . \tag{5.32}$$

It is easy to see that such a b.s. only exists in the $NS$-$NS$ and $R$-$R$ sectors; in what follows, we will label them by $\delta = \frac{1}{2}$ and $\delta = 0$ respectively. Furthermore, at first sight there are two states in each one of the two sectors labelled by $\eta = \pm 1$; however, the GSO projection to be considered below will allow for a linear combination of them to survive. In terms of modes, (5.32) is equivalent to

$$\left(b^M_m - i\,\eta\; M^M{}_N\; \tilde{b}^N_{-m}\right)\; |B^\psi; \eta >= 0 \quad , \quad \forall\, m \in \mathbf{Z}_\delta \quad . \tag{5.33}$$

The solution is [10]

$$\begin{aligned} |B^\psi; \eta_> \;&=\; \prod_{m \in \mathbf{Z}^+_\delta}^{\infty} \mathbf{e}^{i\,\eta\, b^M_{-m}\, M_{MN}\, \tilde{b}^N_{-m}}\; |B^\psi; \eta >_0 \\ |B^\psi; \eta >_0 \;&=\; \begin{cases} |0> & , \quad \delta = \frac{1}{2} \\ |N >= N^{\Lambda\bar{\Lambda}}\; |\Lambda\tilde{\Lambda}> & , \quad \delta = 0 \end{cases} \quad , \end{aligned} \tag{5.36}$$

where, from (5.33) with $m = 0$, it follows that the matrix $N$ must satisfy

$$\Gamma^M\; N = i\,\eta\; M^M{}_N\; \Gamma_{11}\; N\; \Gamma^{N t} \quad . \tag{5.37}$$

[10]The state $|0>$ is the usual vacuum in the NS-NS sector, defined by

$$b^M_m\, |0 >= \tilde{b}^M_m |0 >= 0 \quad , \quad m \;\in\; \mathbf{Z}^+_{\frac{1}{2}} \quad , \tag{5.34}$$

while $|\Lambda\tilde{\Lambda} >\equiv |\Lambda > \otimes |\tilde{\Lambda} >$ is the R-R vacuum obeying

$$\begin{aligned} b^M_m\, |\Lambda\tilde{\Lambda} > \;&=\; \tilde{b}^M_m |\Lambda\tilde{\Lambda} >= 0 \quad , \quad m \;\in\; \mathbf{Z}^+ \\ b^M_0\, |\Lambda\tilde{\Lambda} > \;&=\; \frac{1}{\sqrt{2}}\, \Gamma^{M\,\Omega}{}_\Lambda\, |\Omega\tilde{\Lambda} > \quad ; \quad \tilde{b}^M_0\, |\Lambda\tilde{\Lambda} >= \frac{1}{\sqrt{2}}\, {\Gamma_{11}}^\Omega{}_\Lambda\, \Gamma^{M\tilde{\Omega}}{}_{\tilde{\Lambda}}\, |\Omega\tilde{\Omega} > \quad . \end{aligned} \tag{5.35}$$

The solution is unique up to normalization; if

$$\begin{array}{rcl} A_\pm\,\Gamma^M\,{A_\pm}^{-1} &=& \pm\Gamma^{Mt} \qquad\Longrightarrow\quad A_- = A_+\,\Gamma_{11} = i\,\Gamma_0 \\ U^{-1}\,\Gamma^M\,U &=& {S^{-1M}}_N\,\Gamma^N \quad\Longrightarrow\quad U = S(S^{-1}) \quad , \end{array} \tag{5.38}$$

we can take it to be

$$\begin{array}{rcl} N &=& U\,\Gamma^{01\ldots p}\,\gamma_\pm\,{A_\pm}^{-1} = N^*\,\Gamma_{11} \\ \gamma_\pm &\equiv& \dfrac{1\pm i\,\eta\,\Gamma_{11}}{1\pm i\,\eta} = \gamma_\mp\,\Gamma_{11} \quad , \end{array} \tag{5.39}$$

where $S(S^{-1})$ is the spinor representation of the element $S^{-1}\in SO(1,p)$. Explicitly for the system under consideration,

$$S(S_{0,\pi}) = S\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\,\pi\,\nu^{(1)}_{0,\pi}\,\sigma_2} & 0 & 0 \\ 0 & 0 & e^{i\,\pi\,\nu^{(2)}_{0,\pi}\,\sigma_2} & 0 \\ 0 & 0 & 0 & 1_5 \end{pmatrix}\right) = e^{\frac{\pi}{2}\,\nu^{(1)}_{0,\pi}\,\Gamma_{12}+\frac{\pi}{2}\,\nu^{(2)}_{0,\pi}\,\Gamma_{34}} \quad . \tag{5.40}$$

The hermiticity conditions $\psi^{M\dagger}_m = \psi^M_{-m}$ , $\tilde\psi^{M\dagger}_m = \tilde\psi^M_{-m}$ , for any $m\in\mathbf{Z}$ fix, almost completely (up to normalization), the scalar product to be

$$\begin{array}{rclcl} <0|0> &=& 1 & , & \delta = \dfrac{1}{2} \\ <\Lambda\tilde\Lambda|\Omega\tilde\Omega> &=& g_{\Lambda\Omega}\;\tilde g_{\tilde\Lambda\tilde\Omega} & , & \delta = 0 \quad , \end{array} \tag{5.41}$$

where, in the R-R sector, the $m=0$ conditions $\psi^{M\dagger}_0 = \psi^M_0$ , $\tilde\psi^{M\dagger}_0 = \tilde\psi^M_0$ , impose

$$\left.\begin{array}{rcl} g\,\Gamma^M\,g^{-1} &=& \Gamma^{M\dagger} \\ g\,\Gamma_{11} &=& \pm\Gamma_{11}\,g \\ \tilde g\,\Gamma^M\,\tilde g^{-1} &=& \pm\Gamma^{M\dagger} \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcccccl} g &=& -i\,C_+ &=& \Gamma_0\,\Gamma_{11} &=& g^\dagger \\ \tilde g &=& C_- &=& i\,\Gamma_0 &=& \tilde g^\dagger \end{array}\right. \quad . \tag{5.42}$$

With these definitions we get

$$\begin{array}{rcl} A^\psi(z,\tilde z;\eta,\eta') &\equiv& <B'^\psi;\eta'|\,z^{L^\psi_0-\frac{5}{24}}\;\bar z^{\tilde L^\psi_0-\frac{5}{24}}\;|B^\psi;\eta> \\ &=& |z|^{\frac{5}{6}(1-3\delta)}\displaystyle\prod_{m=1}^{\infty}\det\left(1_{10}+\eta\,\eta'\,|z|^{2(m-\delta)}\,M\,M'^{-1}\right)\,A^\psi_0 \\ A^\psi_0 &\equiv& {}_0<B'_\psi;\eta'|B_\psi;\eta>_0 = \left\{\begin{array}{lcl} 1 & , & \delta=\frac{1}{2} \\ -32\,\eta\,\cos\pi\nu^{(1)}_-\,\cos\pi\nu^{(2)}_-\,\delta_{\eta\eta',1} & , & \delta = 0 \end{array}\right. . \end{array} \tag{5.43}$$

<u>$\beta$-$\gamma$ super-ghost boundary state</u>

According to (C.2) the open string b.c. of a $\lambda=\frac{3}{2}$ commuting, $\beta$-$\gamma$ system are

$$
\begin{array}{rcl}
\tilde{\beta}(\tau,0) & = & \eta_0\, \beta(\tau,0) \quad ; \qquad \tilde{\beta}(\tau,\pi) = \eta_\pi\, \beta(\tau,\pi) \\
\beta(\tau - iT,\sigma) & \sim & \eta_+\, \beta(\tau,\sigma) \quad ; \quad \tilde{\beta}(\tau - iT,\sigma) \sim \eta_-\, \tilde{\beta}(\tau,\sigma) \quad ,
\end{array} \tag{5.44}
$$

and analogously for $\gamma, \tilde{\gamma}$. The phases $\eta_0 \equiv \eta, \eta_\pi$ and $\eta_+, \eta_-$ must be identified with those introduced in (5.2) and (5.29) respectively, because they follow the superconformal transformation parameter modding. With the help of (5.13) the b.c. (5.44) translate, in the closed channel, into the following defining conditions for the b.s. $|B^{\beta\gamma};\eta>$

$$
\begin{array}{rclcl}
\left(\hat{\beta}(0,\hat{\sigma}) + i\,\eta\,\hat{\tilde{\beta}}(0,\hat{\sigma})\right)|B^{\beta\gamma};\eta> & = & 0 & \longleftrightarrow & \left(\hat{\beta}_m + i\,\eta\,\hat{\tilde{\beta}}_{-m}\right)|B^{\beta\gamma};\eta>= 0 \\
\left(\hat{\gamma}(0,\hat{\sigma}) + i\,\eta\,\hat{\tilde{\gamma}}(0,\hat{\sigma})\right)|B^{\beta\gamma};\eta> & = & 0 & \longleftrightarrow & \left(\hat{\gamma}_m + i\,\eta\,\hat{\tilde{\gamma}}_{-m}\right)|B^{\beta\gamma};\eta>= 0 \quad ,
\end{array} \tag{5.45}
$$

for any $m \in \mathbf{Z}_\delta$. We will restrict ourselves to $\tilde{\pi}_0 = 1 - \pi_0$ , the "soaking up" anomaly condition, in which case (5.45) is solved by

$$
|B^{\beta\gamma};\eta>_{\pi_0,\tilde{\pi}_0} = \prod_{m\geq\pi_0} e^{i\,\eta\,\gamma_{-m}\,\tilde{\beta}_{-m}} \prod_{m\geq\tilde{\pi}_0} e^{-i\,\eta\,\beta_{-m}\,\tilde{\gamma}_{-m}}\,|0>_{\pi_0,\tilde{\pi}_0} \tag{5.46}
$$

where $|0>_{\pi_0,\tilde{\pi}_0}$ is defined at left and right as in (C.6). However, there is a very important difference with the anticommuting ghosts; in that case, in view of (C.12), we took, with no loss of generality $p_0 = \tilde{p}_0 = 0$. But, due to the commuting character of the $\beta$-$\gamma$ system, there exists no such identification, i.e., each pair $(\pi_0, \tilde{\pi}_0)$ defines a different representation of the superghost algebra, the so called "pictures", denoted commonly as $(-\frac{1}{2} - \pi_0, -\frac{1}{2} - \tilde{\pi}_0)$. Therefore, the vacuum, as well as the physical (BRST invariant) operators, must be referred to a particular picture, the different pictures being related by the "picture changing operation" of FMS [17]. We will not dwell into details about these facts, but just restrict ourselves to consider the tools necessary to define and compute the amplitudes. The hermiticity properties

$$
{\beta_m}^\dagger = -\beta_{-m}\ ,\ {\gamma_m}^\dagger = \gamma_{-m}\ ,\ {\tilde{\beta}_m}^\dagger = -\tilde{\beta}_{-m}\ ,\ {\tilde{\gamma}_m}^\dagger = \tilde{\gamma}_{-m} \tag{5.47}
$$

determine the scalar product, to be defined just by imposing

$$
{}_{\pi_0',\tilde{\pi}_0'}<0|0>_{\pi_0,\tilde{\pi}_0} \equiv \delta_{\pi_0',1-\pi_0}\,\delta_{\tilde{\pi}_0',1-\tilde{\pi}_0} \quad . \tag{5.48}
$$

We get for the amplitude in the $(-\frac{1}{2} - \pi_0, -\frac{3}{2} + \pi_0)$ picture

$$
\begin{array}{rcl}
A^{\beta\gamma}(z,\tilde{z};\eta,\eta') & \equiv & {}_{1-\pi_0,\pi_0}< B^{\beta\gamma};\eta'|\, z^{L_0^{\beta\gamma} - \frac{11}{24}}\ \tilde{z}^{\tilde{L}_0^{\beta\gamma} - \frac{11}{24}}\ |B^{\beta\gamma};\eta>_{\pi_0,1-\pi_0} \\
& = & |z|^{\frac{1}{6}(3\delta - 1)} \displaystyle\prod_{m=1}^{\infty} \left(1 + \eta\,\eta'\,|z|^{2(m-\delta)}\right)^{-2} \left\{ \begin{array}{ll} (1 + \eta\,\eta')^{-1} & ,\ \delta = 0 \\ (\eta\,\eta')^{\pi_0 - \frac{1}{2}} & ,\ \delta = \frac{1}{2} \end{array} \right. \\
& = & (\eta\,\eta')^{\pi_0 - \frac{1}{2}} \left(Z_{2b}^{1-2\delta}(it)|_{e^{i2\pi b} = \eta\eta',|z| = e^{-\pi t}}\right)^{-1} \quad .
\end{array} \tag{5.49}
$$

Now, we are ready to construct the amplitude in the closed channel. According to (5.15), the bosonic sector gives

$$
\begin{aligned}
A^{(b)}_{closed}(it) &\equiv A^X(z,\tilde{z})\; A^{bc}(z,\tilde{z})|_{z=\bar{z}=e^{-\pi t}} \\
&= i\, N_{B'}{}^*\; N_B\; V_5\; \frac{4\,\sin\pi\bar{\nu}^{(1)}\;\sin\pi\bar{\nu}^{(2)}}{(2\pi^2\alpha')^{\frac{5}{2}}}\; \frac{e^{-\frac{T_s\,\Delta y^2}{t}}}{t^{\frac{5}{2}}\,\eta(it)^4}\; \left(Z^1_{1+2\bar{\nu}^{(1)}}(it)\; Z^1_{1+2\bar{\nu}^{(2)}}(it)\right)^{-1} \quad .
\end{aligned}
\tag{5.50}
$$

In the fermionic sector, we must sum both contributions from the NS-NS and R-R sectors, this last one with a "minus" sign coming from the charge of the anti D-brane. Furthermore, we must project GSO in each one, i.e. we must insert the operator [3]

$$
P_{GSO} \equiv \frac{1+(-)^F}{2}\;\frac{1+(-)^{(p+1)(1-2\delta)}\;(-)^{\tilde{F}}}{2}
\tag{5.51}
$$

in the computation of the traces, where $F = F^\Psi + F^{\beta\gamma}$ and $\tilde{F} = \tilde{F}^\Psi + \tilde{F}^{\beta\gamma}$ are the left and right spinor number operators and $p$ is even (odd) in the type IIA (IIB) theory. Then [11]

$$
\begin{aligned}
A^{(f)}_{closed}(it) &\equiv A^{(f);GSO}_{NS-NS}(it) - A^{(f);GSO}_{R-R}(it) \\
A^{(f);GSO}_{NS-NS}(it) &\equiv A^{\psi;GSO}(z,\tilde{z};\eta,\eta')\; A^{\beta\gamma;GSO}(z,\tilde{z};\eta,\eta')\;|_{\eta=\eta'=1} \\
&= \frac{1}{2}\left(Z^0_0(it)^2\; Z^0_{2\nu^{(1)}_-}(it)\; Z^0_{2\nu^{(2)}_-}(it) - Z^0_1(it)^2\; Z^0_{1+2\nu^{(1)}_-}(it)\; Z^0_{1+2\nu^{(2)}_-}(it)\right) \\
A^{(f);GSO}_{R-R}(it) &\equiv A^{\psi;GSO}(z,\tilde{z};\eta,\eta')\; A^{\beta\gamma;GSO}(z,\tilde{z};\eta,\eta')\;|_{\eta=\eta'=1} \\
&= -\frac{1}{2}\; Z^1_0(it)^2\; Z^1_{2\nu^{(1)}_-}(it)\; Z^1_{2\nu^{(2)}_-}(it) \quad .
\end{aligned}
\tag{5.53}
$$

It follows that, if we normalize the b.s. according to

$$
N_B \equiv \left(\frac{(1+{b^{(1)}}^2)\;(1+{b^{(2)}}^2)}{16\,\pi\,\alpha'}\right)^{\frac{1}{2}} \quad ,
\tag{5.54}
$$

from (5.7), (5.10), (5.50), (5.53) and using the modular properties (D.2), the following identities hold

$$
\begin{aligned}
A^{(b)}_{open}(it^{-1}) &= sign(b^{(1)}\,b^{(2)})\;\pi\,\alpha'\,t\;A^{(b)}_{closed}(it) \\
A^{(f)}_{open}(it^{-1}) &= sign(b^{(1)}\,b^{(2)})\;A^{(f)}_{closed}(it) \quad .
\end{aligned}
\tag{5.55}
$$

It is then straightforward to prove that (5.14) coincides exactly with (5.11).

[11] 8The definition of such amplitudes is equivalent to considering the GSO-projected b.s. [18]

$$
|B>_{GSO} \equiv P_{GSO}|B;+> = \frac{1}{2}\;\left(|B;+> + (-)^{[\pi_0]+1+p(1-2\delta)}|B;->\right)
\tag{5.52}
$$

### 5.3 No force conditions and supersymmetry

To write the amplitude (5.11) just obtained from both channels in a convenient way, we can use the *second addition theorem* for theta functions [21] which states that the following quartic identity

$$\prod_{i=1}^{4} \vartheta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (\nu_i;\tau) = \frac{1}{2} \sum_{s_1,s_2=0,\frac{1}{2}} e^{-i4\pi a_1 s_2} \prod_{i=1}^{4} \vartheta \begin{bmatrix} m_i + s_1 \\ n_i + s_2 \end{bmatrix} (\epsilon_i;\tau) \tag{5.56}$$

holds, where

$$\begin{aligned} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_4 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_4 \end{pmatrix} \\ \begin{pmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes 1_2 \begin{pmatrix} \begin{bmatrix} m_1 \\ n_1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} m_4 \\ n_4 \end{bmatrix} \end{pmatrix} \quad . \end{aligned} \tag{5.57}$$

The famous "abstruse" identity is a special case of it. By choosing the arguments $\epsilon_i = 0$ and the spin structures $m_1 = -\nu_-^{(1)}$ , $n_1 = 1$ , $m_2 = -\nu_-^{(2)}$ and the other zero, we get

$$\frac{1}{2} (\ldots)_{(5.11)} = e^{i\pi(\nu_-^{(1)} - \nu_-^{(2)})} \left( Z_1^{-\nu_-^{(1)} - \nu_-^{(2)}}(it)\, Z_1^{-\nu_-^{(1)} + \nu_-^{(2)}}(it) \right)^2 \quad . \tag{5.58}$$

By virtue of the identity $Z_1^1(it) \equiv 0$, this equation shows that the $D4$-$\bar{D}4$ amplitude is identically zero iff

$$|\nu_-^{(1)}| + |\nu_-^{(2)}| = 1 \quad \longleftrightarrow \quad |b^{(1)}\, b^{(2)}| = 1 \quad , \tag{5.59}$$

which is exactly the condition (2.6) founded for SUSY to hold!

# 6 Conclusions and perspectives

In this paper we have studied a $D4$-$\bar{D}4$ system with non zero magnetic fields in the world-volumes of the branes in the weak coupling regime, and we have shown that it becomes stable and supersymmetric for values of the gauge fields that coincide with those conjectured from the weak coupling, low energy approach defined by the Dirac-Born-Infeld world-volume effective field theory on a brane. We think these results provide further evidence about the existence of the system. In particular, a solution of $D = 10$ SUGRA IIA describing the long distance fields such brane-antibrane system supports should exist, maybe as a certain limit of a more general solution, a kind of "four dimensional supertube", much as it happens with the $D2$-$\bar{D}2$ and the supertube. Finding such solution is certainly a very interesting open problem.

Moreover, it is natural to ask for the effective five dimensional world-volume field theory of the system. A $U(1)$ gauge field $A$ ($\bar{A}$) and five scalars live on the brane (anti-brane), that together with the eight fermionic degrees of freedom fill a vector multiplet of the $D = 5$, $SUSY_1$ algebra (remember that the system preserves 8 supercharges). Furthermore, the two "ex-tachyons" in (4.3), (4.4) are charged under $A^- \equiv A - \bar{A}$ [23]; then, the spectrum analysis of Section 4 reveals the existence of a hypermultiplet of such algebra, coming from the inter-branes excitations and charged under $A^-$. Finally, the commutators among the coordinate fields along the longitudinal directions of the brane-antibrane system (see Appendix B) [12]

$$[Z^{(a)}(\tau,\sigma); Z^{(b)\dagger}(\tau,\sigma')] = 2\,\delta^{ab} \begin{cases} \Theta^{(a)}(\sigma) = \pi\,\alpha' \;\frac{1+b_0^{(a)}\,b_\pi^{(a)}}{b_\pi^{(a)}-b_0^{(a)}}\,\frac{1-b_\sigma^{(a)2}}{1+b_\sigma^{(a)2}} & , \quad \sigma=\sigma'=0,\pi \\ 0 & , \quad otherwise \end{cases} \tag{6.2}$$

signal, in the context of the paper, the non-commutative character of the effective field theory under consideration [24]. More precisely, the obvious relations $b_0^{(a)} = \pm\, b_\pi^{(a)} \equiv b^{(a)}$ yields the well defined non-commutative parameters

$$\Theta^{(a)} = \frac{\pi\,\alpha'}{2}\,\frac{(1-b^{(a)2})^2}{b^{(a)}\,(1+b^{(a)2})} \quad . \tag{6.3}$$

Therefore, we are led to conjecture that the effective field theory is a five dimensional non-commutative $U(1)\times U(1)$ gauge theory, coupled to one matter hypermultiplet charged under one of the $U(1)'s$, with non zero non-commutative parameters in the spatial planes (12) and (34), defined in (6.3) for $a=1$ and $a=2$ respectively. However, when SUSY holds, we have $b^{(1)}b^{(2)} = \pm 1$ and, then, $\Theta^{(1)} = \pm\Theta^{(2)}$ follows. For the special case $|b^{(a)}| = 1$ (or what is the same, $|\nu^{(a)}| = \frac{1}{2}$) there is a naive enhancement of the space-time symmetry from $SO(2)\times SO(2)$ to $U(2)$, as can be checked from the spectrum and, moreover, the non-commutative parameters are null. Does this mean that the effective gauge field theory becomes a commutative one? In any case, the kind of SUSY gauge theory that emerges merits further investigation, maybe along the lines of the boundary string field theory approach [26].

As a last comment, it is possible that more general configurations of gauge fields give rise to tachyon-free brane-antibrane systems preserving some supersymmetries. In particular, we could consider gauge fields in the brane and antibrane which were not necessary parallel as it was recently pursued in reference [27] for $p = 2$. A general analysis for $p \geq 2$, even though it becomes harder, is also a subject for future work.

[12] In its evaluation the following identities valid for $\nu \notin \mathbf{Z}$ are required (see page 46 in [25])

$$\begin{aligned} \sum_{n\in\mathbf{Z}} \frac{\sin n\,\theta}{n+\nu} &= \begin{cases} \pi\,\frac{\sin((2m+1)\pi-\theta)\nu}{\sin\pi\nu} & , \quad m2\pi < \theta < (m+1)2\pi \\ 0 & , \quad \theta = m2\pi \end{cases} \\ \sum_{n\in\mathbf{Z}} \frac{\cos n\,\theta}{n+\nu} &= \pi\,\frac{\cos((2m+1)\pi-\theta)\nu}{\sin\pi\nu} \quad , \quad m2\pi \leq \theta \leq (m+1)2\pi \quad . \end{aligned} \tag{6.1}$$

# A Conventions

We review shortly free field theories on the strip $\Sigma = \{\sigma^1 \equiv \sigma \in [0,\pi], \sigma^0 \equiv \tau \in \mathbf{R}\}$ with superconformal symmetry. We adopt the following conventions. The Minkowskian metric in two dimensions is

$$\eta = \eta_{\mu\nu}\, d\sigma^\mu\, d\sigma^\nu = -d^2\tau + d^2\sigma = -d\sigma^+\, d\sigma^- \quad , \quad \sigma^\pm = \tau \pm \sigma \quad , \quad 2\,\partial_\pm = \partial_\tau \pm \partial_\sigma \quad , \tag{A.1}$$

while the two-dimensional gamma matrices in a Majorana-Weyl basis are

$$\gamma^0 = -i\sigma_2 \;\;,\;\; \gamma^1 = \sigma_1 \;\;,\;\; \gamma^2 = i\gamma^0 = \sigma_2 \;\;,\;\; \gamma_3 = i\gamma^1\gamma^2 = -\sigma_3 \quad , \quad \{\gamma_\mu ; \gamma_\nu\} = 2\,\eta_{\mu\nu} \;\;, \tag{A.2}$$

being $\frac{1}{2}\gamma_{01} = \frac{1}{2}\sigma_3$ the standard Lorentz generator in this spinor representation. We write a generic (Dirac) spinor as

$$\psi \equiv \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = A^{-1} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \quad , \tag{A.3}$$

where we use $A = \sigma_1, A\,\gamma^\mu\,A^{-1} = +\gamma^{\mu t}$, to low (and $A^{-1}$ to raise) spinor indices. As usual $\bar\psi \equiv \psi^\dagger C = i(\psi_+^* \; - \psi_-^*)$ with $C = \sigma_2$ , $C\gamma^\mu C^{-1} = -\gamma^{\mu\dagger}$ the charge conjugation matrix, while the conjugate spinor is defined by $\psi^c \equiv D\psi^*$ where $D^{-1}\,\gamma^\mu\,D = \gamma^{\mu *}$. In the representation chosen, where the $\gamma^\mu$'s are real, we can take $D = 1$ and then a Majorana spinor which, by definition, verifies the reality constraint $\psi^c = \psi$, is just a real one $\psi_\pm^* = \psi_\pm$. We will be using the scales $T_s{}^{-1} = 2\,\pi\,\alpha' = \pi\, l^2$ and the notation $\mathbf{Z}_\nu \equiv \mathbf{Z} + \nu = \{m + \nu\,,\, m \in \mathbf{Z}\}$ , $\mathbf{Z}'_\nu = \mathbf{Z}_\nu - \{0\}$ with $\nu \in [0,1)$.

To begin with, we consider the action $S[X,\psi] = S_0^X[X] + S^\psi[\psi]$ with $X, \psi$ real and

$$\begin{aligned} S_0^X[X] &= -\frac{T_s}{2}\int_\Sigma d^2\sigma\;\eta^{\mu\nu}\,\partial_\mu X\,\partial_\nu X = 2\,T_s\int_\Sigma d^2\sigma\;\partial_+ X\,\partial_- X \\ S^\psi[\psi] &= -\frac{T_s}{2}\int_\Sigma d^2\sigma\;\bar\psi\,\gamma^\mu\,\partial_\mu\psi = i\,T_s\int_\Sigma d^2\sigma\;(\psi_+\,\partial_-\psi_+ + \psi_-\,\partial_+\psi_-) \;\;. \end{aligned} \tag{A.4}$$

Formally, the action is invariant under conformal SUSY transformations with real Grassmann parameters $\epsilon^\pm = \epsilon^\pm(\sigma^\pm)$

$$\begin{aligned} \delta_\epsilon X &= \bar\epsilon\,\psi = -i\,(\epsilon^+\psi_+ - \epsilon^-\psi_-) \\ \delta_\epsilon \psi &= \partial_\mu X\,\gamma^\mu\,\epsilon = \begin{pmatrix} -2\,\partial_- X\,\epsilon^- \\ 2\,\partial_+ X\,\epsilon^+ \end{pmatrix} \quad . \end{aligned} \tag{A.5}$$

The equations of motion derived from (A.4) are solved by

$$\partial_\pm X = F_\pm(\sigma^\pm) \quad , \quad \psi_\pm = f_\pm(\sigma^\pm) \quad , \tag{A.6}$$

for arbitrary functions $F_\pm, f_\pm$ , and the cancellation of the boundary terms

$$
\begin{aligned}
\delta S_0^X|_{b.t.} &= -T_s \int d\tau \; \delta X \; \partial_\sigma X|_{\sigma=0}^{\sigma=\pi} \\
\delta S^\psi|_{b.t.} &= i \, \frac{T_s}{2} \int d\tau \; (\delta\psi_+ \; \psi_+ - \delta\psi_- \; \psi_-)|_{\sigma=0}^{\sigma=\pi}
\end{aligned} \tag{A.7}
$$

in the variation of the action imposes the boundary conditions (b.c.) on the fields as well as the form of the surviving supersymmetry. It is clear that fermions must in general satisfy the following b.c.

$$
\psi_-|_{\sigma=0} = \delta_0 \; \psi_+|_{\sigma=0} \quad , \quad \psi_-|_{\sigma=\pi} = \delta_\pi \; \psi_+|_{\sigma=\pi} \quad . \tag{A.8}
$$

For Majoranas the phases are simply signs, $|\delta_0| = |\delta_\pi| = 1$; the general solution is

$$
\psi_\pm(\sigma_\pm) = \left. \begin{array}{c} 1 \\ \delta_0 \end{array} \right\} \frac{l}{\sqrt{2}} \sum_{r \in \mathbf{Z}_\nu} b_r \; e^{-ir\sigma^\pm} \quad , \quad \nu = \left\{ \begin{array}{ccc} 0 & , & \delta_\pi \, \delta_0 = +1 \\ \frac{1}{2} & , & \delta_\pi \, \delta_0 = -1 \end{array} \right. \tag{A.9}
$$

from which we obtain

$$
b_r = \frac{1}{\sqrt{2}\pi l} \int_0^\pi d\sigma \; \left( e^{ir\sigma^+} \; \psi_+(\sigma_+) + \delta_0 \; e^{ir\sigma^-} \; \psi_-(\sigma_-) \right) \quad . \tag{A.10}
$$

By definition in Ramond (R) sectors bosons and fermions have equal modding while, in Neveu-Schwarz (NS) sectors, the modding differs by a half-integer. So, they will differ according to the four possible choices for the boson b.c. to be considered.

## A.1 NN boundary conditions

Neumann b.c. are considered at both ends,

$$
\partial_\sigma X|_{\sigma=0} = \partial_\sigma X|_{\sigma=\pi} = 0 \quad . \tag{A.11}
$$

In this case, the general solution for the boson field is

$$
\begin{aligned}
X(\tau,\sigma) &= x + l^2 \, p \; \tau + i \, l \sum_{m \in \mathbf{Z}'} \frac{\alpha_m}{m} \; e^{-im\tau} \; \cos m \, \sigma \\
\partial_\pm X(\tau,\sigma) &= \frac{l}{2} \sum_{m \in \mathbf{Z}} \alpha_m \; e^{-im\sigma^\pm} \quad , \quad \alpha_0 \equiv l \, p
\end{aligned} \tag{A.12}
$$

while, for the fermions, compatibility with the superconformal symmetry (A.5) yields the phases $\delta_0 = 1$ and $\delta_\pi = \pm 1$ in the R/NS sectors.

## A.2 DD boundary conditions

Dirichlet b.c. are considered at both ends,

$$\partial_\tau X|_{\sigma=0} = \partial_\tau X|_{\sigma=\pi} = 0 \quad . \tag{A.13}$$

In this case, the general solution for the boson field is ($\Delta x \equiv x_\pi - x_0$)

$$\begin{aligned} X(\tau,\sigma) &= x_0 + \frac{\Delta x}{\pi}\,\sigma + l \sum_{m\in\mathbf{Z}'} \frac{\alpha_m}{m}\, e^{-im\tau}\, \sin m\,\sigma \\ \partial_\pm X(\tau,\sigma) &= \pm\frac{l}{2} \sum_{m\in\mathbf{Z}} \alpha_m\, e^{-im\sigma^\pm} \quad , \quad \alpha_0 \equiv \frac{\Delta x}{\pi\, l} \equiv l\, p \end{aligned} \tag{A.14}$$

while, for the fermions, compatibility with SUSY transformations yields the phases $\delta_0 = -1$ and $\delta_\pi = \pm 1$ in the NS/R sectors.

## A.3 ND boundary conditions

In this case, Neumman b.c. are taken at one end and Diritchlet b.c. at the other one,

$$\partial_\sigma X|_{\sigma=0} = \partial_\tau X|_{\sigma=\pi} = 0 \quad . \tag{A.15}$$

The general solution for the boson field is

$$X(\tau,\sigma) = x_\pi + i\, l \sum_{r\in\mathbf{Z}_{\frac{1}{2}}} \frac{\alpha_r}{r}\, e^{-ir\tau}\, \cos r\,\sigma \quad , \quad \partial_\pm X(\tau,\sigma) = \frac{l}{2} \sum_{r\in\mathbf{Z}_{\frac{1}{2}}} \alpha_r\, e^{-ir\sigma^\pm} \tag{A.16}$$

while, for the fermions, compatibility with SUSY transformations yields the phases $\delta_0 = +1$ and $\delta_\pi = \pm 1$ in the NS/R sectors.

## A.4 DN boundary conditions

This case is the same as the ND one, with the ends interchanged,

$$\partial_\tau X|_{\sigma=0} = \partial_\sigma X|_{\sigma=\pi} = 0 \quad . \tag{A.17}$$

The general solution for the boson field is

$$X(\tau,\sigma) = x_0 + l \sum_{r\in\mathbf{Z}_{\frac{1}{2}}} \frac{\alpha_r}{r}\, e^{-ir\tau}\, \sin r\,\sigma \quad , \quad \partial_\pm X(\tau,\sigma) = \pm\frac{l}{2} \sum_{r\in\mathbf{Z}_{\frac{1}{2}}} \alpha_r\, e^{-ir\sigma^\pm} \tag{A.18}$$

while, for the fermions, compatibility with SUSY transformations yields the phases $\delta_0 = -1$ and $\delta_\pi = \pm 1$ in the R/NS sectors.

In all the four cases, the SUSY parameters are characterized by the conditions

$$\epsilon^+(\tau) = -\epsilon^-(\tau) \equiv \epsilon(\tau) \quad , \quad \epsilon(\tau+2\pi) = \pm\epsilon(\tau) \quad R/NS\ sector \quad . \tag{A.19}$$

In particular the surviving global SUSY charge (that certainly lives in the R sector) is the combination $Q_+ + Q_- \sim G_0$ (see (A.24)).

## A.5 Quantization and $N=1$ superconformal algebra

From the action (A.4) we read the canonical (anti) commutation relations

$$\begin{aligned} [X(\tau,\sigma);\partial_\tau X(\tau,\sigma')] &= \frac{i}{T_s}\,\delta(\sigma-\sigma') \quad\longleftrightarrow\quad [\alpha_m;\alpha_n] = m\,\delta_{m+n,0} \\ \{\psi_\pm(\tau,\sigma);\psi_\pm(\tau,\sigma')\} &= \frac{1}{T_s}\,\delta(\sigma-\sigma') \quad\longleftrightarrow\quad \{b_r;b_s\} = \delta_{r+s,0} \quad . \end{aligned} \tag{A.20}$$

Reality conditions translate into ${\alpha_{-m}}^\dagger = \alpha_m$ , ${b_{-r}}^\dagger = b_r$. Quantum-mechanically, a normal ordering prescription is needed to define quantum operators; this is done from the usual representation of the canonical conmutation relations (A.20) [13]

$$|0> \ : \quad \alpha_m|0> = b_m|0> = 0\ ,\ m>0 \tag{A.21}$$

by putting at right (left) the destruction (creation) operators. The (traceless) energy-momentum tensor is defined by $T_{\mu\nu} \equiv -\frac{2}{T_s}\frac{\delta S}{\delta g^{\mu\nu}}|_{g=\eta}$; its non zero components for bosons and fermions are, respectively,

$$\begin{aligned} T^X_{\pm\pm}(\sigma^\pm) &= \partial_\pm X\,\partial_\pm X \equiv \alpha' \sum_{m\in\mathbf{Z}} L^X_m\, e^{-im\sigma^\pm} \\ L^X_m &= \frac{1}{2}\sum_n :\alpha_{m-n}\,\alpha_n: + \Delta^X_0\,\delta_{m,0} \quad , \quad [L^X_m;\alpha_n] = -n\,\alpha_{m+n} \\ T^\psi_{\pm\pm}(\sigma^\pm) &= \frac{i}{2}\,\psi_\pm\,\partial_\pm\psi_\pm \equiv \alpha' \sum_{m\in\mathbf{Z}} L^\psi_m\, e^{-im\sigma^\pm} \\ L^\psi_m &= \frac{1}{2}\sum_r (r-\frac{m}{2})\ :b_{m-r}\,b_r: + \Delta^\psi_0\,\delta_{m,0} \quad , \quad [L^\psi_m;b_r] = -(r+\frac{m}{2})\,b_{m+r} \quad . \end{aligned} \tag{A.22}$$

The Virasoro generators $\{L^X_m\}$ and $\{L^\psi_m\}$ obey the algebra in (A.26) with $c^X = 1$ and $c^\psi = \frac{1}{2}$ respectively, provided that the conformal dimensions of the vacuum states are [14]

$$\Delta^X_0 = \left\{ \begin{array}{lcl} -\frac{1}{24}+\frac{1}{24} = & 0 & \mathrm{NN}\ ,\ \mathrm{DD} \\ +\frac{1}{48}+\frac{1}{24} = & \frac{1}{16} & \mathrm{ND}\ ,\ \mathrm{DN} \end{array}\right. \quad , \quad \Delta^\psi_0 = \left\{ \begin{array}{lcl} -\frac{1}{48}+\frac{1}{48} = & 0 & \mathrm{AP} \\ +\frac{1}{24}+\frac{1}{48} = & \frac{1}{16} & \mathrm{P} \end{array}\right. \quad . \tag{A.23}$$

Similarly, the fermionic supercurrent defined by $G_\mu \equiv -\frac{1}{2T_s}\frac{\delta S}{\delta\chi^\mu}|_{\chi=0}$ , where $\chi^\mu$ is the gravitino field, has the components

$$\begin{aligned} G_\pm(\sigma^\pm) &= \psi_\pm\,\partial_\pm X \equiv \frac{\alpha'}{\sqrt{2}} \sum_{r\in\mathbf{Z}_\delta} G_r\, e^{-ir\sigma^\pm} \quad , \quad \delta = \left\{ \begin{array}{ll} 0\ , & R \\ \frac{1}{2}\ , & NS \end{array}\right. \\ G_r &= \sum_m \alpha_m\, b_{r-m} \quad , \quad \{G_r;b_s\} = \alpha_{r+s} \quad , \quad [G_r;\alpha_m] = -m\,b_{r+m} \quad . \end{aligned} \tag{A.24}$$

[13] For periodic (P) fermions, if we identify $|0>\equiv|->$. Then, $b_0|\pm> = \frac{1}{\sqrt{2}}|\mp>$ .

[14] Alternatively they can be computed by using the Hurwitz zeta-function $\xi(s,x) \equiv \sum_{k=0}^\infty (k+x)^{-s}$ (particularly useful cases are $\xi(-1,x) = -\frac{1}{12}(1+6x^2-6x)$ and $\xi(-2,x) = -\frac{1}{6}x(1-x)(1-2x)$) to regulate the infinite sums and then adding the Casimir energy $\frac{c}{24}$, see [2].

In the NN or DD (with $x_0 = x_\pi$) cases, the NS vacuum defined in (A.21) is the unique $Osp(1,2)$ (SUSY extension of $SL(2,\mathbf{R})$) invariant one,

$$L_m\,|0>_{NS} = G_r\,|0>_{NS} = 0 \quad , \quad m \geq -1 \ , \ r \geq -\frac{1}{2} \quad . \tag{A.25}$$

This is not so with ND/DN b.c. (or DD with $\Delta x \neq 0$) because $L_{-1}$ does not annihilate it; for example, $L^X_{-1}|0>_{NS} = \frac{1}{2}\alpha_{-\frac{1}{2}}{}^2|0>_{NS}$ ($\frac{\Delta x}{\pi l}\,\alpha_{-1}|0>_{NS}$ ).

As we saw, the combined system in each case present two sectors, the NS sector with opposite modding and the R sector with equal modding. The standard form of the $N=1$ superconformal algebra

$$\begin{aligned}
[L_m; L_n] &= (m-n)\,L_{m+n} + \frac{c}{12}\,m\,(m^2-1)\,\delta_{m+n,0} \\
[L_m; G_r] &= (\frac{m}{2} - r)\,G_{m+r} \\
\{G_r; G_s\} &= 2\,L_{r+s} + \frac{c}{12}\,(4\,r^2 - 1)\,\delta_{r+s,0} \quad ,
\end{aligned} \tag{A.26}$$

where $c = \frac{3}{2}$ is the central charge of the system, follows, with

$$\Delta_0^{(R)} = \frac{1}{16} \quad , \quad \Delta_0^{(NS)} = \begin{cases} 0 & \text{NN, DD} \\ \frac{1}{8} & \text{ND, DN} \end{cases} \quad . \tag{A.27}$$

# B A system with mixed boundary conditions

Let us take a complex scalar field $Z = X^1 + i\,X^2$, together with a Dirac fermion $\Psi = \psi^1 + i\,\psi^2$ with $X^i, \psi^i$ real, and consider the action $S = S_0^Z + S^\Psi + S_b^Z$, where

$$\begin{aligned}
S_0^Z &= -\frac{T_s}{2}\int_\Sigma d^2\sigma\;\eta^{\mu\nu}\;\partial_\mu Z^*\;\partial_\nu Z = S_0^{X^1} + S_0^{X^2} \\
S^\Psi &= -\frac{T_s}{4}\int_\Sigma d^2\sigma\;(\bar\Psi\;\gamma^\mu\,\partial_\mu\Psi + h.h.) = S^{\psi^1} + S^{\psi^2} \\
S_b^Z &= \frac{T_s}{2}\int d\tau\;\epsilon_{ij}\;\Big(b_\pi\;X^i(\tau,\pi)\partial_\tau X^j(\tau,\pi) - b_0\;X^i(\tau,0)\partial_\tau X^j(\tau,0)\Big) \quad .
\end{aligned} \tag{B.1}$$

The boundary term $S_b$ can be interpreted as the coupling, with unit charge, of the ends of the string to a gauge field $A$ of strength $F \equiv dA = T_s\,\tilde b(\sigma)\;dX^1 \wedge dX^2$ , where $\tilde b(0) = b_0$ , $\tilde b(\pi) = b_\pi$ and zero otherwise.

The bosonic sector

The boundary conditions that follow from (B.1) are

$$(\partial_\sigma + i\,b_0\,\partial_\tau)Z(\tau,\sigma)|_{\sigma=0} = (\partial_\sigma + i\,b_\pi\,\partial_\tau)Z(\tau,\sigma)|_{\sigma=\pi} = 0 \quad . \tag{B.2}$$

It will be convenient to introduce the following notation

$$1 + i\,b_0 \equiv \sqrt{1+b_0{}^2}\;e^{i\varphi_0} \quad , \quad \varphi_0 \equiv \frac{\pi}{2}\,\nu_0 \quad ; \quad 1 + i\,b_\pi \equiv \sqrt{1+b_\pi{}^2}\;e^{i\varphi_\pi} \quad , \quad \varphi_\pi \equiv \frac{\pi}{2}\,\nu_\pi \ , \tag{B.3}$$

where $\nu_0\,(\nu_\pi)$ varies from $-1$ to $+1$ as $b_0\,(b_\pi)$ goes from $-\infty$ to $+\infty$. We also introduce

$$\nu_\pm \equiv \frac{1}{2}\,(\nu_\pi \pm \nu_0) \quad , \qquad \bar{\nu} = \begin{cases} \nu_- & , \qquad 0 \le \nu_- < 1 \\ \nu_- + 1 & , \quad -1 < \nu_- < 0 \end{cases} \quad . \tag{B.4}$$

The general solution with the boundary conditions (B.2) can be written

$$\begin{aligned} Z(\tau,\sigma) &= z + \sqrt{2}\,l\,A_0\;\phi_0(\tau,\sigma) + i\,\sqrt{2}\,l\sum_{r\in\mathbf{Z}'_{\bar{\nu}}} \frac{A_r}{r}\;\phi_r(\tau,\sigma) \\ \partial_\pm Z(\tau,\sigma) &= \frac{l}{\sqrt{2}}\sum_{r\in\mathbf{Z}_{\bar{\nu}}} A_r\;e^{-ir\sigma^\pm \mp i\varphi_0} \quad , \end{aligned} \tag{B.5}$$

where we have introduced the functions

$$\begin{aligned} \phi_0(\tau,\sigma) &= \cos\varphi_0\;\tau - i\,\sin\varphi_0\,\sigma \\ \phi_r(\tau,\sigma) &= e^{-ir\tau}\;\cos(r\,\sigma + \varphi_0) \quad , \quad r \in \mathbf{Z}'_{\bar{\nu}} \end{aligned} \tag{B.6}$$

and it is understood that $A_0 \equiv 0$ unless $\bar{\nu} = 0$.

Let us introduce, at fixed time, the pairing

$$(\phi_1;\phi_2) \equiv \frac{1}{2i}\int_0^\pi d\sigma\;(\phi_1^*\,\partial_\tau\phi_2 - \partial_\tau\phi_1^*\;\phi_2) + \frac{b_\pi}{2}\;\phi_1^*\;\phi_2|_{\sigma=\pi} - \frac{b_0}{2}\phi_1^*\;\phi_2|_{\sigma=0} \quad . \tag{B.7}$$

It is easy to prove that, for $\phi_1, \phi_2$ of the type (B.5), it does not depend on $\tau$. In particular, the following orthogonality relations hold

$$\begin{aligned} (\phi_r;\phi_s) &= \delta_{r,s}\begin{cases} -\frac{\pi}{2}\,r & , \quad r \neq 0 \\ \frac{\pi^2}{2}\,b_0 & , \quad r = 0 \end{cases} \\ (1;\phi_r) &= \frac{\pi}{2\,i}\;\sec\varphi_0\;\delta_{r,0} \\ (1;1) &= \frac{\sin\pi\nu_-}{\cos\pi\nu_+ + \cos\pi\nu_-} = \frac{1}{2}\,(b_\pi - b_0) \quad . \end{aligned} \tag{B.8}$$

From them, it follows that

$$\begin{aligned} (1;Z) &= (1;1)\;z - \sqrt{2}\,l\;(\phi_0;1)\;A_0 \\ (\phi_0;Z) &= (\phi_0;1)\;z + \sqrt{2}\,l\;(\phi_0;\phi_0)\;A_0 \\ (\phi_r;Z) &= \frac{\pi\,l}{\sqrt{2}\,i}\;A_r \quad , \quad r \in \mathbf{Z}'_{\bar{\nu}} \quad . \end{aligned} \tag{B.9}$$

Then, the bulk commutation relations implied by (B.1)

$$[Z(\tau,\sigma);\partial_\tau Z(\tau,\sigma')] = i\,\frac{2}{T_s}\;\delta(\sigma-\sigma') \tag{B.10}$$

yield, from (B.9), the non trivial commutation relations [15]

[15] An useful relation is the following one; if $f_i = (\phi_i;Z)$ , $i = 1,2$ , then $[f_1;{f_2}^\dagger] = -\frac{1}{T_s}\;(\phi_1;\phi_2)$.

$$
\begin{aligned}
[z;z^\dagger] &= 2\,\theta \quad \longleftrightarrow \quad [x^i;x^j] = i\,\theta\,\epsilon^{ij} \quad , \quad z \equiv x^1 + i\,x^2 \\
[A_r;{A_s}^\dagger] &= r\,\delta_{r,s} \quad , \quad [z;{A_0}^\dagger] = i\,\sqrt{2}\,l\,\cos\varphi_0 \quad ,
\end{aligned} \tag{B.11}
$$

where the non-commutative parameter $\theta$ is given by

$$
\theta = \begin{cases} -(2\,T_s\,(1;1))^{-1} = \frac{1}{T_s}\,\frac{1}{b_0 - b_\pi} & , \quad \bar{\nu} \neq 0 \\ \frac{\sin \pi\nu_0}{2\,T_s} = \frac{1}{T_s}\,\frac{b_0}{1+{b_0}^2} & , \quad \bar{\nu} = 0 \end{cases} \quad . \tag{B.12}
$$

The fermionic sector

From the action for the Dirac spinor in (B.1) with the boundary conditions as in (A.8), but now with $\delta_0 \equiv e^{i\phi_0}$ and $\delta_\pi \equiv e^{i\phi_\pi}$ arbitrary phases, the general solution is given by

$$
\Psi_\pm(\sigma_\pm) = l \sum_{r \in \mathbf{Z}_\nu} B_r\; e^{-ir\sigma^\pm \mp i\frac{\phi_0}{2}} \quad , \quad \delta_\pi\,\delta_0^* = e^{i(\phi_\pi - \phi_0)} \equiv e^{i2\pi\nu} \quad . \tag{B.13}
$$

However, consistency with $N = 1$ superconformal symmetry transformations [16]

$$
\begin{aligned}
\delta_\epsilon\, Z &= \bar{\epsilon}\;\Psi = -i\,(\epsilon^+\Psi_+ - \epsilon^-\Psi_-) \\
\delta_\epsilon \Psi &= \partial_\mu Z\;\gamma^\mu\;\epsilon = \begin{pmatrix} -2\,\partial_- Z\;\epsilon^- \\ 2\,\partial_+ Z\;\epsilon^+ \end{pmatrix} \quad ,
\end{aligned} \tag{B.14}
$$

where the parameter $\epsilon$ is Majorana, fixes the phases to be

$$
\delta_0 = e^{i\pi\nu_0} \quad , \quad \delta_\pi = \epsilon\; e^{i\pi\nu_\pi} \quad , \quad \delta_\pi\;\delta_0^* = \epsilon\; e^{i2\pi\nu_-} \quad , \tag{B.15}
$$

with $\epsilon$ an arbitrary sign, as well as the form of the SUSY parameters $\epsilon^\pm(\sigma^\pm)$

$$
\epsilon^+(\tau) = -\epsilon^-(\tau) \equiv \epsilon(\tau) \quad , \quad \epsilon(\tau + 2\pi) = \epsilon\;\epsilon(\tau) \quad . \tag{B.16}
$$

It is easy to see that we can identify the sector with $\epsilon = -1$ as the NS sector and that with $\epsilon = +1$ with the R one, just by realizing, from (B.13) and (B.15), that, in the last case, the fermionic modding coincides with the bosonic one ($\nu = \bar{\nu}$) while, in the first one, the modding differs by a half-integer ($\nu = \bar{\nu} \pm \frac{1}{2}$ if $0 \leq \bar{\nu} < \frac{1}{2} \,/\, \frac{1}{2} \leq \bar{\nu} < 1$); furthermore, from (B.16) the right periodicity for the SUSY parameter follows.

Now, by using

$$
B_r = \frac{1}{2\,\pi\,l} \int_0^\pi d\sigma\; \left(e^{i(r\sigma^+ + \varphi_0)}\;\Psi_+(\sigma^+) + e^{i(r\sigma^- - \varphi_0)}\;\Psi_-(\sigma^-)\right) \quad , \tag{B.17}
$$

the canonical anti-commutation relations read

$$
\{\Psi(\tau,\sigma);\Psi^\dagger(\tau,\sigma')\} = \frac{2}{T_s}\;\delta(\sigma - \sigma') \quad \longleftrightarrow \quad \{B_r;B_s^\dagger\} = \delta_{r,s} \quad . \tag{B.18}
$$

[16] Equivalently, it is possible to derive the b.c. from a fermionic boundary term added to the action (B.1), see for example [19].

A reference state, w.r.t. the normal ordering will be understood, is defined by the conditions

$$\begin{array}{rcl} |0>_{r_0}:\ A_r\,|0>_{r_0} &=& 0\ ,\ r\geq 0\ \ ,\ \ {A_r}^\dagger\,|0>_{r_0}=0\ ,\ \ r\leq 0 \\ |0>_{r_0}:\ B_r\,|0>_{r_0} &=& 0\ ,\ r\geq r_0\ \ ,\ \ {B_r}^\dagger\,|0>_{r_0}=0\ ,\ \ r< r_0 \end{array} \tag{B.19}$$

for some $r_0\in\mathbf{Z}_\nu$.

## B.1 Conserved currents and $N=2$ superconformal algebra

Two conserved currents associated with this system can be defined. The first one is just the momentum associated to the translations $Z\to Z+c$ , $c\in\mathbb{C}$, and it is defined by [17]

$$\begin{array}{rcl} P_i^\mu(\tau,\sigma) &\equiv& -T_s\,(\partial^\mu X_i(\tau,\sigma)-\epsilon_{ij}\,\epsilon^{\mu\nu}\,\partial_\nu(\tilde b(\sigma)\,X^j(\tau,\sigma)))\quad,\quad \epsilon_{\tau\sigma}=\epsilon^{\sigma\tau}\equiv+1 \\ p_z &\equiv& p_1+i\,p_2=\displaystyle\int_0^\pi d\sigma\;P_z^\tau(\tau,\sigma)=\left\{\begin{array}{ll}\frac{1}{i\,\theta}\,z & ,\quad \bar\nu\neq 0\\ \frac{\sqrt2}{l\,\cos\varphi_0}\,A_0 & ,\quad \bar\nu=0\end{array}\right. \quad . \end{array} \tag{B.20}$$

They are, as they should, the canonical conjugate variables to the center of mass coordinates,

$$[x^i;x^j]=i\,\theta\,\epsilon^{ij}\quad,\quad [x^i;p_j]=i\,\delta^i_j\quad,\quad [p_i;p_j]=0\quad . \tag{B.21}$$

We can get a representation of this zero mode algebra (for $\nu_-=0$) in momentum space by defining the action of the operators as

$$\hat p_i\longrightarrow p_i\quad,\quad \hat x^i\longrightarrow i\,\frac{\partial}{\partial p_i}-\theta\,\epsilon^{ij}\,p_j \tag{B.22}$$

or, in a more standard way, we can introduce canonical variables defined by

$$\begin{array}{rcl} q^1 &=& c\,x^1-s\,p_2\quad,\quad k_1=c\,x^2+s\,p_1 \\ q^2 &=& s\,x^1+c\,p_2\quad,\quad k_2=-s\,x^2+c\,p_1\quad , \end{array} \tag{B.23}$$

where $c\equiv\frac{1}{\sqrt{1+{\theta_2}^2}}$ , $s\equiv\frac{\theta_2}{\sqrt{1+{\theta_2}^2}}$ , $\theta_{1/2}\equiv\sqrt{1+\frac{\theta^2}{4}}\pm\frac{\theta}{2}>0$ . They verify

$$\begin{array}{rcl} [q^i;q^j]=[k_i;k_j]=0 &,& [q^i;k_j]=i\,\theta_i\,\delta^i_j \\ dx^1\wedge dx^2\wedge dp_1\wedge dp_2 &=& dq^1\wedge dq^2\wedge dk_1\wedge dk_2\quad . \end{array} \tag{B.24}$$

In view of the relation $\theta_1\,\theta_2=1$ , the phase space volume remains invariant, an important fact in computing the zero mode contribution to the partition function. On the other hand, when $\bar\nu\neq 0$ we have just the $x$-variables, introducing a factor of $(2\,\pi\,\theta)^{-1}$ in such computation.

[17] The reader notes that for $|\nu_0|=1$ ($|b_0|\to\infty$) *or* $|\nu_\pi|=1$ ($|b_\pi|\to\infty$) it does not exist, in agreement with the fact that (B.2) becomes Diritchlet boundary conditions in at least one end.

The second conserved current $J_\mu \equiv J^Z_\mu + J^\Psi_\mu$ leads to the conserved charge $Q = Q^Z + Q^\Psi$ that generates the $U(1)$ rotations $Z \rightarrow e^{-i\epsilon}\, Z$ , $\Psi \rightarrow e^{-i\epsilon}\, \Psi$ , and it is defined by

$$
\begin{aligned}
J^Z_\mu(\tau,\sigma) &\equiv T_s\, Im\,(Z^*(\tau,\sigma)\, \partial_\mu Z(\tau,\sigma)) + \frac{T_s}{2}\, \epsilon_{\mu\nu}\, \partial^\nu (\tilde{b}(\sigma)\, Z^*(\tau,\sigma)\, Z(\tau,\sigma)) \\
&= X^2 P^\mu_1 - X^1 P^\mu_2 \\
Q^Z &\equiv \int_0^\pi d\sigma\; J^{Z\,\tau}(\tau,\sigma) = \sum_{r\in \mathbf{Z}'_{\nu_-}} \frac{:A^\dagger_r\, A_r:}{r} + \begin{cases} \frac{\vec{x}^2}{2\,\theta} & ,\ \nu_- \neq 0 \\ -\frac{\theta}{2}\, \vec{p}^2 - x^1\, p_2 + x^2\, p_1 & ,\ \nu_- = 0 \end{cases} \\
J^\Psi_\mu(\tau,\sigma) &\equiv i\,\frac{T_s}{2}\, \bar{\Psi}\, \gamma_\mu\, \Psi \\
Q^\Psi &\equiv \int_0^\pi d\sigma\; J^{\Psi\,\tau}(\tau,\sigma) = \frac{T_s}{2} \int_0^\pi d\sigma\; (\Psi^\dagger_+\, \Psi_+ + \Psi^\dagger_-\, \Psi_-) = \sum_{r\in\mathbf{Z}_\nu} :B^\dagger_r\, B_r: \quad .
\end{aligned}
\tag{B.25}
$$

While the bosonic part of the $U(1)$ current suggested us to introduce the pairing (B.7), the fermionic part can be extended to a holomorphic current,

$$
\begin{aligned}
J_\pm(\sigma_\pm) &\equiv i\,\frac{T_s}{2}\, \bar{\Psi}\, \gamma_\pm\, \Psi = -\frac{T_s}{2}\, \Psi^\dagger_\pm\, \Psi_\pm \equiv -\frac{1}{2\pi} \sum_{m\in\mathbf{Z}} J_m\, e^{-im\sigma^\pm} \quad , \quad J^\dagger_{-m} = J_m \\
J_m &= \sum_{r\in\mathbf{Z}_\nu} :B^\dagger_r\, B_{m+r}: + q_0\, \delta_{m,0} \quad , \quad [J_m ; B_r] = -B_{m+r}
\end{aligned}
\tag{B.26}
$$

whose zero component is essentially the global charge defined above, $J_0 = Q^\Psi + q_0$, where $q_0 \equiv r_0 - \frac{1}{2}$ is the $U(1)$ charge of $|0>_{r_0}$. This current is just the $U(1)$ current of the $N=2$ superconformal algebra the system realizes, the fermion number current. In fact, we can introduce the *complex* fermionic currents

$$
\begin{aligned}
G_\pm(\sigma^\pm) &\equiv \frac{1}{2}\, \Psi^*_\pm\, \partial_\pm Z \equiv \frac{\alpha'}{\sqrt{2}} \sum_{r\in\mathbf{Z}_\delta} G^{(+)}_r\, e^{-ir\sigma^\pm} \quad , \quad \delta = \begin{cases} 0 & ,\ R \\ \frac{1}{2} & ,\ NS \end{cases} \\
G^{(+)}_r &\equiv \sum_{s\in\mathbf{Z}_{\bar{\nu}}} A_s\, B^\dagger_{s-r} \quad , \quad [G^{(+)}_r ; A^\dagger_s] = s\, B^\dagger_{s-r} \quad , \quad \{G^{(+)}_r ; B_s\} = A_{r+s} \quad .
\end{aligned}
\tag{B.27}
$$

The modes of the hermitian conjugate $J_\pm(\sigma^\pm)^\dagger$ are introduced in the same way, with the result $G^{(-)}_r = G^{(+)\dagger}_{-r}$. Furthermore, the modes of the bosonic and fermionic energy-momentum tensors

$$
\begin{aligned}
T^Z_{\pm\pm} &\equiv \partial_\pm Z^\dagger\, \partial_\pm Z \equiv \alpha' \sum_{m\in\mathbf{Z}} L^Z_m\, e^{-im\sigma^\pm} \\
L^Z_m &= \sum_{r\in\mathbf{Z}_{\bar{\nu}}} :A^\dagger_r\, A_{r+m}: + \Delta^Z_0\, \delta_{m,0} \quad , \quad [L^Z_m ; A_r] = -r\, A_{m+r} \\
T^\Psi_{\pm\pm} &\equiv \frac{i}{4}\, (\Psi^\dagger_\pm\, \partial_\pm \Psi_\pm + \Psi_\pm\, \partial_\pm \Psi^\dagger_\pm) \equiv \alpha' \sum_{m\in\mathbf{Z}} L^\Psi_m\, e^{-im\sigma^\pm} \\
L^\Psi_m &= \sum_{r\in\mathbf{Z}_\nu} (r + \frac{m}{2})\; :B^\dagger_r\, B_{m+r}: + \Delta^\Psi_0\, \delta_{m,0} \quad , \quad [L^\Psi_m ; B_r] = -(r + \frac{m}{2})\, B_{m+r}
\end{aligned}
\tag{B.28}
$$

verify the standard Virasoro algebra in (A.26) with $c^Z = 2$ , $c^\Psi = 1$, respectively, and the constants

$$\Delta_0^Z = \frac{1}{8} - \frac{1}{2}(\bar\nu - \frac{1}{2})^2 \quad , \quad \Delta_0^\Psi = \frac{1}{2}(r_0 - \frac{1}{2})^2 \quad . \tag{B.29}$$

Thus, the modes of the mixed system, $L_m = L_m^Z + L_m^\Psi$ obey the Virasoro algebra with $c = 3$ and a conformal dimension of the vacuum given by

$$\Delta = \Delta_0^Z + \Delta_0^\Psi = \frac{1}{8} + \frac{1}{2}(r_0 - \bar\nu)(r_0 + \bar\nu - 1) \quad . \tag{B.30}$$

This energy-momentum tensor, together with the currents (B.26), (B.27), generate the $N = 2$ superconformal algebra which completes the Virasoro one with

$$\begin{aligned}
\{G_r^{(+)}; G_s^{(-)}\} &= L_{r+s} + \frac{r-s}{2} J_{r+s} + \frac{c}{24}(4r^2 - 1)\,\delta_{r+s,0} \\
[J_m; J_n] &= \frac{c}{3}\, m\, \delta_{m+n,0} \\
[L_m; J_n] &= -n\, J_{m+n} \\
[L_m; G_r^{(\pm)}] &= (\frac{m}{2} - r)\, G_{m+r}^{(\pm)} \\
[J_m; G_r^{(\pm)}] &= \pm\, G_{m+r}^{(\pm)} \quad .
\end{aligned} \tag{B.31}$$

It is easy to check that the current $G_\pm + G_\pm^\dagger$, with modes $G_r = G_r^{(+)} + G_r^{(-)}$, closes in the $N = 1$ superconformal algebra (A.26), which is an important fact because it is this hermitian current that enters in the super Virasoro constraints.

All this is true for any choice of $r_0 \in \mathbf{Z}_\nu$ in (B.19). The choice

$$r_0 = \bar\nu + \delta \tag{B.32}$$

defines states with minimal conformal dimension, i.e. vacuums, except in the NS sector, when $\frac{1}{2} \leq \bar\nu < 1$, where such a state is $B_{\bar\nu - \frac{1}{2}}|0>_{\bar\nu + \frac{1}{2}} = |0>_{\bar\nu - \frac{1}{2}}$ . In any case, due to the identifications

$$|0>_{r_0+n} \equiv \begin{cases} B_{r_0+n} \dots B_{r_0-1}|0>_{r_0} & , \quad n \in \mathbf{Z}^- \\ B^\dagger_{r_0+n-1} \dots B^\dagger_{r_0}\,|0>_{r_0} & , \quad n \in \mathbf{Z}^+ \end{cases} \quad , \tag{B.33}$$

all the states $\{|0>_{r_0+n}\ ,\ n \in \mathbf{Z}\}$ are in the representation defined by $|0>_{r_0}$; so, with no loss of generality, we will take for $r_0$ the value (B.32). It is worth to noting that any of these vacua is $Osp(1,2)$ invariant unless $\bar\nu = 0$, in which case it belongs to the NS sector. This follows, for example, from $L_{-1}^Z|0> = A_{\bar\nu}^\dagger\, A_{\bar\nu - 1}\,|0>$ , and $L_{-1}^\Psi|0>_{r_0} = (r_0 - \frac{1}{2})B_{r_0}^\dagger\, B_{r_0-1}|0>_{r_0}$. The superconformal invariant vacuum can be written in the bosonized form of the theory; we do not dwell in such details here [3].

# C Ghost systems

Ghost systems are generically defined as theories of fields that obey commutation relation (statistics) opposite to the usual ones assigned according to their behaviour under Lorentz transformations (spin) by the CPT theorem, i.e., spin integer fields are anti-commuting, while half-integer spin fields are commuting. They naturally appear as a Fadeed-Popov representation of determinants that come from fixing gauge symmetries. In superstring theories, two dimensional reparameterization invariance gives rise to the anticommuting $\lambda = 2$ $b$-$c$ system, while the gauge fixing of the local world-sheet SUSY gives rise to the commuting $\lambda = \frac{3}{2}$ $\beta$-$\gamma$ system. Furthermore, an anticommuting $\lambda = 1$ $\eta$-$\xi$ system is needed in the process of bosonization of the last ones.

## C.1 Anticommuting $b$-$c$ systems

Let us consider a pair of anticommuting, completely symmetric, and traceless tensors $c \in \tau_0^{\lambda-1}$ and $b \in \tau_\lambda^0$ with $\lambda \in \mathbf{Z}$; in the conformal gauge they have components ($c \equiv c^{+\cdots+}, \tilde{c} \equiv c^{-\cdots-}$) and ($b \equiv b_{+\ldots+}, \tilde{b} \equiv b_{-\ldots-}$) respectively, with action

$$S^{bc} = i\, T_s \int_\Sigma d^2\sigma \, \left( c\, \partial_- b + \tilde{c}\, \partial_+ \tilde{b} \right) \quad . \tag{C.1}$$

The equations of motion and boundary conditions to be considered

$$\begin{aligned} \partial_- b &= \partial_- c = \partial_+ \tilde{b} = \partial_+ \tilde{c} = 0 \\ \tilde{b}|_{\sigma=0,\pi} &= e^{i\gamma_{0,\pi}}\, b|_{\sigma=0,\pi} \quad , \quad \tilde{c}|_{\sigma=0,\pi} = e^{-i\gamma_{0,\pi}}\, c|_{\sigma=0,\pi} \end{aligned} \tag{C.2}$$

yield the $\epsilon_0 \equiv \frac{\gamma_\pi - \gamma_0}{2\,\pi}$ modded expansions

$$\begin{aligned} c(\sigma^+) &= l \sum_{p\in \mathbf{Z}_{\epsilon_0}} c_{-p}\, e^{ip\sigma^+ + i\frac{\gamma_0}{2}} \quad , \quad \tilde{c}(\sigma^-) = l \sum_{p\in \mathbf{Z}_{\epsilon_0}} c_{-p}\, e^{ip\sigma^- - i\frac{\gamma_0}{2}} \\ b(\sigma^+) &= l \sum_{p\in \mathbf{Z}_{\epsilon_0}} b_p\, e^{-ip\sigma^+ - i\frac{\gamma_0}{2}} \quad , \quad \tilde{b}(\sigma^-) = l \sum_{p\in \mathbf{Z}_{\epsilon_0}} b_p\, e^{-ip\sigma^- + i\frac{\gamma_0}{2}} \end{aligned} \tag{C.3}$$

while, with the help of

$$\begin{aligned} c_{-p} &= \frac{1}{2\pi l} \int_0^\pi d\sigma \, \left( e^{-i\frac{\gamma_0}{2} - ip\sigma^+}\, c(\sigma^+) + e^{i\frac{\gamma_0}{2} - ip\sigma^-}\, \tilde{c}(\sigma^-) \right) \\ b_p &= \frac{1}{2\pi l} \int_0^\pi d\sigma \, \left( e^{i\frac{\gamma_0}{2} + ip\sigma^+}\, b(\sigma^+) + e^{-i\frac{\gamma_0}{2} + ip\sigma^-}\, \tilde{b}(\sigma^-) \right) \quad , \end{aligned} \tag{C.4}$$

the canonical anti-commutation relations read

$$\{ b(\tau,\sigma); c(\tau,\sigma') \} = \frac{2}{T_s}\, \delta(\sigma - \sigma') \quad \longleftrightarrow \quad \{ c_{-p}\,; b_q \} = \delta_{p,q} \quad . \tag{C.5}$$

We remark that the case $\epsilon_0 = 0$ is the one relevant to string theory, because this is the modding of the reparameterization parameters.

A ghost reference state (loosely called "vacuum") is defined by the conditions

$$b_p\,|0>_{p_0}=0 \quad , \quad p\geq p_0 \quad ; \quad c_{-p}\,|0>_{p_0}=0 \quad , \quad p<p_0 \quad , \tag{C.6}$$

where $p_0\in\mathbf{Z}_{\epsilon_0}$. With respect to it, we define the components of the energy-momentum tensor (we omit the right moving expressions)

$$\begin{aligned}
T(\sigma^+) &= \frac{1-\lambda}{2\,i}\,\partial_+ b\,c : -\frac{\lambda}{2\,i}\,b\,\partial_+ c :\equiv \alpha' \sum_{m\in\mathbf{Z}} L_m^{bc}\,e^{-im\sigma^+}\\
L_m^{bc} &= \sum_{p\in\mathbf{Z}_{\epsilon_0}} ((\lambda-1)\,m-p)\ :b_{m+p}\,c_{-p}: +\delta_{m,0}\,\Delta_0^{bc}\\
[L_m^{bc};b_p] &= ((\lambda-1)\,m-p)\ b_{m+p} \quad , \quad [L_m^{bc};c_{-p}]=(-\lambda\,m+p)\ c_{m-p} \quad .
\end{aligned} \tag{C.7}$$

Moreover, the symmetry under $c\to e^{\lambda}c$ , $b\to e^{-\lambda}\,b$ gives rise to a conserved ghost number current

$$\begin{aligned}
U(\sigma^+) &\equiv -:b\,c:=l^2\sum_{m\in\mathbf{Z}} U_m\,e^{-im\sigma^+} \quad ; \quad U_m\equiv-\sum_{p\in\mathbf{Z}_{\epsilon_0}} :b_{m+p}\,c_{-p}: +u_0\,\delta_{m,0}\\
[U_m;b_p] &= -b_{m+p} \quad , \quad [U_m;c_{-p}]=+c_{m-p} \quad .
\end{aligned} \tag{C.8}$$

The operators $L_m^{bc}$, $U_m$ , satisfy the standard algebra (with $\epsilon=+1$)

$$\begin{aligned}
[L_m^{bc};L_n^{bc}] &= (m-n)\,L_{m+n}^{bc}+\frac{c(\lambda)}{12}\,m\,(m^2-1)\,\delta_{m+n,0}\\
[L_m^{bc};U_n] &= -n\,U_{m+n}+\frac{Q}{2}\,m\,(m+1)\,\delta_{m+n,0}\\
[U_m;U_n] &= \epsilon\,m\,\delta_{m+n,0} \quad ,
\end{aligned} \tag{C.9}$$

where the central charge $c(\lambda)$ and background charge $Q$ are

$$c(\lambda)=-2+12\,\lambda\,(1-\lambda)=1-3\,Q^2 \quad , \quad Q=1-2\,\lambda \quad , \tag{C.10}$$

while the vacuum conformal dimension and ghost charge are determined to be

$$\begin{aligned}
\Delta_0^{bc} &= \frac{1}{2}\,(p_0-\lambda)\,(p_0+\lambda-1)=\frac{1}{2}\,u_0\,(u_0+Q)\\
u_0 &= p_0+\lambda-1 \quad .
\end{aligned} \tag{C.11}$$

We remark that the minimal conformal dimension state ("vacuum") is $|0>_{\epsilon_0}$; only when $\epsilon_0=0$ the state $|+>\equiv c_0\,|0>_0=|0>_1$ is degenerate with $|->\equiv|0>_0$, both with dimension $\Delta_0^{bc}=\frac{\lambda}{2}\,(1-\lambda)$. However, the $SL(2,\mathbf{R})$ invariant vacuum $|1>=|0>_{1-\lambda}$ does not coincide (except when $\lambda=1$) with any of them. In any case, as it happens with the fermions, the identifications

$$|0>_{p_0+m}\equiv\begin{cases} b_{p_0+m}\ldots b_{p_0-1}\,|0>_{p_0} & , \quad m\in\mathbf{Z}^-\\ c_{-p_0-m+1}\ldots c_{-p_0}\,|0>_{p_0} & , \quad m\in\mathbf{Z}^+\end{cases} \tag{C.12}$$

allow us to take $p_0=\epsilon_0$ with no loss of generality.

## C.2 Commuting $\beta - \gamma$ systems

Similarly to what we have just considered, we can take now a pair of spinorial but commuting fields $(\beta, \gamma)$ where, this time, $\lambda$ is half-integer. With the replacements $(b, c)$ by $(\beta, \gamma)$, $(b_p, c_{-p})$ by $(\beta_p, \gamma_{-p})$, etc, things go similarly, so we just point out the main differences. In the first place the commutation relation

$$[\gamma_{-p}; \beta_q] = \delta_{p,q} \quad , \quad p, q \in \mathbf{Z}_{\epsilon_0} \tag{C.13}$$

holds. The cases of interest in superstring theory are $\epsilon_0 = 0$ and $\epsilon_0 = \frac{1}{2}$, which follow the modding of the world-sheet SUGRA transformation parameters in the R and NS sectors respectively. A reference state $|0>_{\pi_0}$ is defined as in (C.6) but, since relations such as (C.12) do not exist, each one of them defines different representations or $u_0 = 1 - \lambda - \pi_0$ "pictures" [18]. For $\lambda = \frac{3}{2}$, for example, we have $u_0 = -\frac{1}{2} - \pi_0$; the vacuum states are the states with $\pi_0 = \frac{1}{2}$ and $\pi_0 = 0$ in the NS and R sectors respectively, the commonly used "$-1$" and "$-\frac{1}{2}$" pictures. It is worth noting that, due to the bosonic character of the operators, the R vacuum state $|0>_0$ is infinitely degenerated (instead of doubly) with the set of states ${\gamma_0}^{m_0}|0>_0$ with $m_0 = 0, 1, 2, \ldots$. On the other hand, the $SL(2, \mathbf{R})$ invariant state is instead identified with the NS state $|0>_{1-\lambda}$, the reference state of the "0" picture.

The parameters and constants of the conformal-ghost algebra (C.9) (with $\epsilon = -1$) are

$$\begin{aligned} c(\lambda) &= 2 - 12\,\lambda\,(1-\lambda) = -1 + 3\,Q^2 \quad , \quad Q = 2\,\lambda - 1 \\ \Delta_0^{\beta\gamma} &= \frac{1}{2}\,(\pi_0 - \lambda)\,(1 - \lambda - \pi_0) = -\frac{1}{2}\,u_0\,(u_0 + Q) \\ u_0 &= 1 - \lambda - \pi_0 \quad . \end{aligned} \tag{C.14}$$

Of particular interest is the combined system of a $\lambda = 2$ $b$-$c$ and $\lambda = \frac{3}{2}$ $\beta$-$\gamma$, present in the superstring. In this case we have

$$c^g = -26 + 11 = -15 \quad , \quad \Delta_0^g = -1 + \left\{ \begin{array}{c} \frac{3}{8} \\ \frac{1}{2} \end{array} \right. = \left\{ \begin{array}{ll} -\frac{5}{8} & , \; if \;\; R \\ -\frac{1}{2} & , \; if \;\; NS \end{array} \right. \quad . \tag{C.15}$$

# D Partition functions

Here we list the relevant partition functions of the various systems considered in precedence. We introduce the variables $q \equiv e^{i2\pi\tau} = e^{-2\pi t}$ and $z \equiv e^{i2\pi b}$, $V_n$ stands for the $\mathbf{R}^n$-volume, the prime in a trace symbol means omitting the zero mode sector and the modular functions $\eta(\tau)$, $\theta_{\alpha\beta}(\nu,\tau)$ are as in chapter 7 of [2] (see i.e. [21] for more),

$$
\begin{aligned}
\eta(\tau) &\equiv q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1-q^m) \\
Z^{2a}_{2b}(\tau) &\equiv \frac{\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(0;\tau)}{\eta(\tau)} = \frac{\vartheta_{2a,2b}(0;\tau)}{\eta(\tau)} \\
&= z^a\, q^{\frac{a^2}{2}-\frac{1}{24}} \prod_{m=1}^{\infty} (1+z\,q^{m-\frac{1}{2}+a})\,(1+z^{-1}\,q^{m-\frac{1}{2}-a})
\end{aligned}
\tag{D.1}
$$

They satisfy the modular properties

$$
\begin{aligned}
\eta(\tau) &= (-i\,\tau)^{-\frac{1}{2}}\,\eta(-\tau^{-1}) = e^{-i\frac{\pi}{12}}\,\eta(\tau+1) \\
\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu;\tau) &= (-i\,\tau)^{-\frac{1}{2}}\,e^{i\pi(2ab-\tau^{-1}\nu^2)}\,\vartheta\begin{bmatrix} b \\ -a \end{bmatrix}(-\tau^{-1}\nu;-\tau^{-1}) \\
&= e^{i\pi a(a+1)}\,\vartheta\begin{bmatrix} a \\ b-a-\frac{1}{2} \end{bmatrix}(\nu;\tau+1) \\
Z^{2a}_{2b}(\tau) &= e^{i2\pi ab}\,Z^{2b}_{-2a}(-\tau^{-1}) = e^{i\pi(a^2+a+\frac{1}{12})}\,Z^{2a}_{2b-2a-1}(\tau+1)
\end{aligned}
\tag{D.2}
$$

We get

$$
\begin{aligned}
Z^X(\tau) &\equiv tr\; q^{L_0^X-\frac{1}{24}} = \begin{cases} V_1\,(2\pi\,l)^{-1}\,t^{-\frac{1}{2}}\,\eta(\tau)^{-1} & ,\quad NN \\ e^{-T_s\,\Delta x^2\,t}\,\eta(\tau)^{-1} & ,\quad DD \\ (Z^0_1(\tau))^{-\frac{1}{2}} & ,\quad ND/DN \end{cases} \\
Z^\psi(\tau) &\equiv tr\; e^{i\pi\beta F}\,q^{L_0^\psi-\frac{1}{48}} = \left(Z^\alpha_\beta(\tau)\right)^{\frac{1}{2}} \quad ,\quad \alpha = \begin{cases} 0 & , \quad NS \\ 1 & , \quad R \end{cases} \quad ,\ \beta = \begin{cases} 0 & , \quad AP \\ 1 & , \quad P \end{cases} \\
Z^{bc}(\tau) &\equiv \begin{cases} tr_{p_0}\; q^{L_0^{bc}-\frac{c(\lambda)}{24}}\,z^{U_0^{bc}-\lambda+\frac{1}{2}} = Z^{2p_0+1}_{2b}(\tau) \\ tr'_{p_0}\; q^{L_0^{bc}-\frac{c(\lambda)}{24}}\,(-)^{U_0^{bc}-\lambda+1} \equiv \frac{Z^1_{2b}(\tau)}{2\cos\pi b}\big|_{b=\frac{1}{2}} = \eta(\tau)^2 \quad , \quad p_0 \in \mathbf{Z} \end{cases} \\
Z^{\beta\gamma}(\tau) &\equiv \begin{cases} e^{i\pi(\pi_0-\frac{1}{2})}\,tr_{\pi_0}\; q^{L_0^{\beta\gamma}-\frac{c(\lambda)}{24}}\,z^{U_0^{\beta\gamma}+\lambda-\frac{1}{2}} = (Z^{1-2\pi_0}_{1-2b}(\tau))^{-1} & , \quad (\pi_0,b) \notin \mathbf{Z}\times\mathbf{Z} \\ tr'_{\pi_0}\; q^{L_0^{\beta\gamma}-\frac{c(\lambda)}{24}} \equiv 2\sin\pi b\,(Z^1_{1-2b}(\tau))^{-1}|_{b=0} = \eta(\tau)^{-2} & , \quad \pi_0 \in \mathbf{Z} \end{cases} \\
Z^Z(\tau) &\equiv tr_{\bar\nu}\; q^{L_0^Z-\frac{1}{12}} = \begin{cases} \frac{V_2}{2\pi\theta}\,e^{i\pi(\frac{1}{2}-\bar\nu)}\,(Z^{1-2\bar\nu}_1(\tau))^{-1} & , \quad \bar\nu \neq 0 \\ \frac{V_2}{8\pi^2\alpha'\cos^2\varphi_0\,t}\,\eta(\tau)^{-2} & , \quad \bar\nu = 0 \end{cases} \\
Z^\Psi(\tau) &\equiv tr_{r_0}\; q^{L_0^\Psi-\frac{1}{24}}\,z^{-J_0^\Psi} = Z^{1-2r_0}_{2b}(\tau)
\end{aligned}
\tag{D.3}
$$

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